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Classical methods for Partial Differential Equations

01. Problem Sheet

Exercise 1:
On the unit ball $\Omega := \{x \in \mathbb{R}^3 : \|x\| \leq 1\}$ consider for $u \in C^2(\Omega)$ the boundary value problem

$$\begin{cases} u_t = \Delta u + f & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

where n is the unit exterior normal to $\partial\Omega$ and $f \in C^\infty(\Omega)$.
Explain why the Neumann boundary condition $\frac{\partial u}{\partial n} = 0$ implies isolation.

Exercise 2:
Let $\Omega := [0, \pi] \times \mathbb{R}$ and consider the initial boundary value problem

$$\begin{cases} u_t = u_{xx}\,, & \text{in } \Omega\,, \\ \frac{\partial u}{\partial x}(0,t) = 0\,, & t \in \mathbb{R}\,, \\ \frac{\partial u}{\partial x}(\pi,t) = 0\,, & t \in \mathbb{R}\,, \\ u(x,0) = f(x)\,, & \text{for } f \in C^\infty([0,\pi])\,. \end{cases}$$

Assume that $\tilde{u} \in C^\infty([0,\pi] \times \mathbb{R})$ is a solution for this problem. Show that

$$\int_0^\pi \tilde{u}(x,t)\,dx = \int_0^\pi f(x)\,dx$$

and compare this result with Exercise 1.

Exercise 3:
Let $f(x) = 3 + \cos(x) + \frac{1}{2}\cos(2x)$. Find functions $a(t)$, $b(t)$, $c(t)$ such that

$$w(x,t) := a(t) + b(t)\cos(x) + c(t)\cos(2x)$$

is a solution of the initial boundary value problem from Exercise 2.

Moreover compute $\lim_{t\to\infty} w(x,t)$. What do you observe?

— **Please turn the page!** —

Exercise 4:
Let $\Omega \subset \mathbb{R}^n$ be a domain. For $u \in C^2(\Omega)$ and $f \in C(\Omega)$ assume that

$$-\Delta u(x) = f(x) \quad \text{in } \Omega\,.$$

Moreover, let $T \in O(n)$ denote an orthogonal $n \times n$ matrix and define $\widetilde{u}(x) := u(Tx)$ and $\widetilde{f}(x) := f(Tx)$ for $x \in \widetilde{\Omega} := \{T^{-1}y \,:\, y \in \Omega\}$.

Show that $\widetilde{u}$ solves

$$-\Delta \widetilde{u}(x) = \widetilde{f}(x) \quad \text{in } \widetilde{\Omega}\,.$$

Hint: Show that the Laplace operator is rotational invariant, i.e. $-\Delta\widetilde{u}(x) = (-\Delta u)(Tx)$ for all $x \in \widetilde{\Omega}$.