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Classical methods for Partial Differential Equations

10. Problem Sheet

Exercise 1:
Prove the following property assumed for the derivation of the solution in Example 5.4.
If $g \in C^2([0,\infty))$ with $g(0) = g''(0) = 0$, then $\widetilde{g} \in C^2(\mathbb{R})$, where

$$\widetilde{g}(x) := \begin{cases} g(x)\,, & x \geq 0\,, \\ -g(-x)\,, & x < 0\,. \end{cases}$$

Exercise 2: (Inhomogeneous 1-dim wave equation with homogeneous initial conditions)
Show the following:
Let $f \in C^1(\overline{\mathbb{R}_+} \times \mathbb{R})$ and $v(x,\sigma,\tau)$ a solution of the homogeneous 1-dim wave equation

$$\begin{cases} v_{tt}(\tau,\sigma,x) = c^2 v_{xx}(\tau,\sigma,x)\,, \quad x \in \mathbb{R}\,,\ \sigma > 0\,, \\ v(\tau,0,x) = 0\,, \quad x \in \mathbb{R}\,, \\ v_t(\tau,0,x) = f(\tau,x)\,, \quad x \in \mathbb{R}\,. \end{cases}$$

where $\tau > 0$ is an arbitrary parameter. Then

$$u^0(t,x) := \int_0^t v(\tau, t-\tau, x)\, d\tau$$

is a solution of the inhomogeneous 1-dim wave equation with homogeneous initial conditions.

Exercise 3: (The Proof of Theorem 5.8)
For $g \in C^3(\mathbb{R}^2)$ and $h \in C^2(\mathbb{R}^2)$ the problem

$$\begin{cases} u_{tt} = \Delta u\,, \quad (t,x) \in (0,\infty) \times \mathbb{R}^2\,, \\ u(0,x) = g(x)\,, \quad x \in \mathbb{R}^2\,, \\ u_t(0,x) = h(x)\,, \quad x \in \mathbb{R}^2\,. \end{cases}$$

has a unique solution in $C^2((0,\infty) \times \mathbb{R}^2) \cap C([0,\infty) \times \mathbb{R}^2)$. This solution is given by

$$u(t,x) = \frac{1}{2\pi t} \int_{B_t(x)} \frac{t h(y) + g(y) + \nabla g(y)(y-x)}{\sqrt{t^2 - |x-y|^2}}\, dy$$

Moreover, for $t > 0$ and all $x_0 \in \mathbb{R}^2$,

$$\lim_{(t,x)\to(0,x_0)} u(t,x) = g(x_0)\,,$$
$$\lim_{(t,x)\to(0,x_0)} u_t(t,x) = h(x_0)\,.$$

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Exercise 4:
Give solutions for the following partial differential equations:

1) For the transport equation ($x \in \mathbb{R}$ and $t > 0$):

$$\textbf{(a)}\ u_t + 5u_x = \sin(x+t)\,, \quad u(0,x) = e^x$$

2) For the inhom. 1-dim wave equation ($x \in \mathbb{R}$ and $t > 0$):

$$\begin{aligned}
\textbf{(b)}\ u_{tt} &= c^2\, u_{xx} + \sin(wx)\,, \quad u(0,x) = u_t(0,x) = 0\,,\\
\textbf{(c)}\ u_{tt} &= 4u_{xx} + \sin(x)\,, \quad u(0,x) = \sin(x)\,,\ u_t(0,x) = 1\,,\\
\textbf{(d)}\ u_{tt} &= c^2\, u_{xx} + \sin(wt)\,, \quad u(0,x) = u_t(0,x) = 0\,,
\end{aligned}$$

3) For the hom. 3-dim wave equation ($x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $t > 0$) with initial conditions:

$$\begin{aligned}
\textbf{(e)}\ u_{tt} &= u_{xx} \quad u(0,x) = e^{x_1}\cos(x_2)\,,\ u_t(0,x) = x_1^2 - x_2^2\,,\\
\textbf{(f)}\ u_{tt} &= u_{xx} \quad u(0,x) = x_1^2 + x_2^2\,,\ u_t(0,x) = 1\,.
\end{aligned}$$

Hint: For 3) use the Ansatz $u(t,x) = \sum_{k=0}^{\infty}\left[\frac{t^{2k}}{(2k)!}\Delta^k u(0,x) + \frac{t^{2k+1}}{(2k+1)!}\Delta^k u_t(0,x)\right]$.

We wish you a merry christmas and all the best for the year 2018 !