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Classical methods for Partial Differential Equations

11. Problem Sheet

Exercise 1:
Let $f,\, g \in \mathcal{S}(\mathbb{R}^n)$. Show the following assertions

a) $f * g \in \mathcal{S}(\mathbb{R}^n)$,

b) $\widecheck{fg} = \check{f} * \check{g}$.

Exercise 2:
Using properties of the Fourier transform derive non-rigourosly the solution of the heat equation

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + f(t,x)\,, & (t,x) \in (0,\infty) \times \mathbb{R}^n\,, \\ u(0,x) = g(x)\,, & x \in \mathbb{R}^n\,, \end{cases}$$

where $f \in \mathcal{S}(\mathbb{R}^{n+1})$, $g \in \mathcal{S}(\mathbb{R}^n)$.

Exercise 3:
Let $g \in \mathcal{S}(\mathbb{R}^n)$ and let

$$u(t,x) = (g * \Phi(t,\cdot))\,(x) \quad , \text{ for } t > 0\,,$$

where $\Phi(t,x)$ is the fundamental solution of the homogenous heat equation

$$\begin{cases} u_t(t,x) = \Delta u(t,x)\,, & (t,x) \in (0,\infty) \times \mathbb{R}^n\,, \\ u(0,x) = g(x)\,, & x \in \mathbb{R}^n\,. \end{cases}$$

Show, using the Fourier transform, that

a) the function u is C^2 for $x \in \mathbb{R}^n$ and $t > 0$ and solves the homogenous heat equation.

b) $u(t,x) \to g(x)$ uniformly in x as $t \to 0$.

c) $\int_{\mathbb{R}^n} |u(t,x) - g(x)|^2 dx \to 0$ as $t \to 0$.

Exercise 4:
Show for $\lambda \in \mathbb{R}$ that there does not exists a non-trivial Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$ such that

$$-\Delta f = \lambda f\,.$$