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Classical methods for Partial Differential Equations

12. Problem Sheet

Exercise 1: (A semi-norm for $\mathcal{S}(\mathbb{R}^k)$)
We define for $n,\, m \in \mathbb{N}$ and $f \in \mathcal{S}(\mathbb{R}^k)$

$$\|f\|_{n,m} := \sup_{x\in\mathbb{R}^k} |x^n f^{(m)}(x)|$$

where $f^{(m)}$ denotes the m-th derivate of f. Show for all $f,\, g \in \mathcal{S}(\mathbb{R}^k)$ and $\alpha \in \mathbb{C}$:

a) $\|f\|_{n,m} \geq 0$,

b) $\|\alpha\, f\|_{n,m} = |\alpha|\|f\|_{n,m}$,

c) $\|f+g\|_{n,m} \leq \|f\|_{n,m} + \|g\|_{n,m}$.

Exercise 2: (A metric for $\mathcal{S}(\mathbb{R}^k)$)
We define a function $d : \mathcal{S}(\mathbb{R}^k) \times \mathcal{S}(\mathbb{R}^k) \to \mathbb{R}$ by

$$d(f,g) := \sum_{n,m=0}^{\infty} 2^{-n-m} \min\left(1, \|f-g\|_{n,m}\right).$$

Show that

a) $d(f,g)$ is well-defined for all $f,\, g \in \mathcal{S}(\mathbb{R}^k)$ and is a metric.

b) with the metric show that the function $\delta : \mathcal{S}(\mathbb{R}^k) \to \mathbb{C}$ given by $\delta(\phi) = \phi(0)$ is linear continuous.

Exercise 3: (Classification of second order PDEs)

a) Classify for all $(x,y) \in \mathbb{R}^2$ the type(s) of the following PDEs:

(1) $u_{xx} + xy\, u_{yy} = 0$

(2) $e^{2x}\, u_{xx} + 2e^{x+y} u_{xy} + e^{2y}\, u_{yy} = 0$

(3) $(x^2-1)u_{xx} + 2xyu_{xy} + (y^2-1)u_{yy} + xu_x + yu_y = 0$

(4) $(x-y)u_{xx} + (xy - y^2 - x + y)u_{xy} + 13u_x - 5e^y u_y + \sinh(xy)\, u = 0$

b) Is the differential operator

$$\frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z \partial x} + \frac{\partial^2 u}{\partial x \partial y}$$

elliptic, parabolic or hyperbolic?