Institute for Analysis
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Classical methods for Partial Differential Equations

13. Problem Sheet

Exercise 1: (Uniqueness of weak limits in Hilbert spaces)
Prove uniqueness of the weak limit in Hilbert spaces.

Exercise 2:
Let

$$\eta(x) = \begin{cases} e^{\frac{1}{|x|^2-1}} \,, & \text{if } |x| < 1\,, \\ 0 \,, & \text{if } |x| \geq 1\,, \end{cases}$$

Prove (or at least sketch the proof) that $\eta \in C_c^\infty(\mathbb{R}^n)$.

Exercise 3: (Minkowski's integral inequality)
Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{C}$ measurable and $1 \leq p < \infty$. Prove that

$$\left[\int \left| \int f(x,y)\, dy\right|^p \mathrm{d}x\right]^{\frac{1}{p}} \leq \int \left[\int |f(x,y)|^p\, dx\right]^{\frac{1}{p}} \mathrm{d}y$$

Conclude the inequality used in the proof of Lemma 9.9

$$\left[\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \eta(v) \int_0^1 |\nabla u(x+\epsilon\, t\, v)|\, dt\, dv\right|^p dx\right]^{\frac{1}{p}} \leq \int_{\mathbb{R}^n} \eta(v) \int_0^1 \left[\int_{\mathbb{R}^n} \left|\nabla u(x+\epsilon\, t\, v)\right|^p dx\right]^{\frac{1}{p}} dt\, dv$$

Exercise 4:
Let $U := (a,b)$ for some a, $b \in \mathbb{R}$ and assume that $u \in C^1(\overline{U}) \cap W_0^{1,p}(U)$, $1 \leq p < \infty$. Prove that $u|_{\partial \overline{U}} = 0$.