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# Classical methods for Partial Differential Equations

## 02. Problem Sheet

**Exercise 1:**
Show that the function $u : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}\,;\, \vec{x} \mapsto \frac{1}{\|\vec{x}\|}$ is harmonic in $\mathbb{R}^3 \setminus \{0\}$.

**Exercise 2:**
Let $\varphi : (0, \infty) \to \mathbb{R}$ be a $C^2$ function and $u : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}\,;\, x \mapsto \varphi(|x|)$ where

$$r = |x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}\,.$$

Show that $u \in C^2(\mathbb{R}^n \setminus \{0\})$ and $\Delta u(x) = \varphi''(r) + \frac{n-1}{r}\varphi'(r)$.

**Exercise 3:**
For $u$, $\varphi$ as in Exercise 3 and $n \geq 2$ show the following equivalence

$$\text{u is harmonic in } \mathbb{R}^n \setminus \{0\} \quad \Longleftrightarrow \quad \begin{cases} \varphi(r) = a + b \log r\,, n = 2 \\ \varphi(r) = a + br^{2-n}\,, n > 2 \end{cases}$$

**Exercise 4:**
Suppose $f \in C^2(\mathbb{R}^3)$ has compact support. Moreover define a function $G_3 : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ by

$$G_3(x) = \frac{1}{|\partial B_1(0)|}|x|^{-1}\,.$$

In the lecture it will be shown that

$$u(x) = G_3 * f := \int_{\mathbb{R}^3} G_3(x-y) f(y) dy$$

is a solution of the Poisson equation $-\Delta u(x) = f(x)$.

Show the following:

1. For a spherical symmetric function $f \in C^2(\mathbb{R}^3)$ that has support inside a ball $B_R(0)$ with $R > 0$ one has

$$V = G_3 * f = \frac{1}{4\pi|x|} \int f(x) dx$$

   for all $x \in \mathbb{R}^3 \setminus B_R(0)$.

2. Give a physical interpretation of this result.