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Classical methods for Partial Differential Equations

04. Problem Sheet

Exercise 1:
For $n \geq 2$ consider a spherical symmetric function $\rho : \mathbb{R}^n \to \mathbb{R}\,;\, y \mapsto \rho(|y|)$. Moreover, assume that $|\gamma_n(x-y)\,\rho(y)|$ is Lebesgue integrable for all $x \in \mathbb{R}^n$. Show *Newtons theorem*, i.e.

$$\int_{\mathbb{R}^n} \gamma_n(x-y)\,\rho(y)\,dy = \gamma_n(x) \int_{|y|<|x|} \rho(y)\,dy + \int_{|y|\geq|x|} \gamma_n(y)\,\rho(y)\,dy\,.$$

Exercise 2:
For $R > 0$ and $x, y \in \mathbb{R}^3$ let an attractive potential V be given by

$$V(x) = -\int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|} dy\,,$$

where $\rho \in C^2(\mathbb{R}^3)$ is constant and nonzero inside the ball $B_R(0)$ and equal to zero outside.

a) Calculate $-\nabla V$.

b) Give a physical interpretation of the result in a).

Exercise 3:
Show the assertion (7) from the proof of Theorem 1.17 stated in the lecture. More precisely, show for $R > 0$

$$\Delta_x\, P(x,y) = 0\,,$$

on $B_R(0) \times \partial B_R(0)$ where $P(x,y) = \frac{R^2-|x|^2}{R\,w_n\,|x-y|^n}$.

Exercise 4:
Let Ω be a bounded domain and $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Moreover, let L be an elliptic differential operator of the form

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} b^i(x) u_{x_i}\,,$$

where the coefficient functions a^{ij} and b^i are continuous in $\overline{\Omega}$.
For $f \in C(\Omega)$ and $g \in C(\partial\Omega)$ show that the boundary value problem

$$\begin{cases} Lu = f\,, & \text{in } \Omega \\ u = g\,, & \text{on } \partial\Omega \end{cases}$$

has at most one solution.