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Classical methods for Partial Differential Equations

05. Problem Sheet

Exercise 1:
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and G be a Green's Function on Ω. Prove that G is unique.

Exercise 2:
Let $U \subset \mathbb{R}^n$ be an open and bounded. Assume that $u \in C^2(U) \cap C(\overline{U})$ such that

$$u^- := -\min(u,0) \neq 0,$$
$$u^+ := \max(u,0) \neq 0.$$

Moreover let $c \geq 0$ in U. Show the following modification of Theorem 2.4.

(i) If $Lu \leq 0$ in U, then $\max_{x\in\overline{U}} u(x) \leq \max_{x\in\partial U} u(x)$

(ii) If $Lu \geq 0$ in U, then $\min_{x\in\overline{U}} u(x) \geq \min_{x\in\partial U} u(x)$

where L is an elliptic differential operator and the coefficient functions a^{ij}, b^i and c are continuous. Moreover a^{ij} is symmetric.

Exercise 3:
Let $U \subset \mathbb{R}^n$ be an open and bounded. u, $v \in C^2(U) \cap C(\overline{U})$ and $c \geq 0$ in U. Moreover let L be an elliptic differential operator as in Exercise 2. Show the following two assertions.

a) If $Lu \leq 0$ and $Lv \geq 0$ in U and $u \leq v$ on ∂U then $u \leq v$ in $\overline{U}$.

b) The boundary value problem

$$\begin{cases} Lu = f\,, & \text{in } U \\ u = g\,, & \text{on } \partial U \end{cases}$$

has at most one solution for $f \in C(U)$ and $g \in C(\partial U)$.

Exercise 4:
Let $U \subset \mathbb{R}^n$ be an open and bounded and $c \geq 0$ in U. Moreover let L be an elliptic differential operator as in Exercise 2. For $g \in C(\partial U)$ let $u \in C^2(U) \cap C(\overline{U})$ be a solution of the boundary value problem

$$\begin{cases} Lu = 0\,, & \text{in } \Omega \\ u = g\,, & \text{on } \partial\Omega \end{cases}$$

Show that

$$\|u\|_\infty \leq \|g\|_\infty\,.$$