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Classical methods for Partial Differential Equations

06. Problem Sheet

Exercise 1:
Let the function $u : \mathbb{R}^n \to \mathbb{R}$ be given by $u(x) = g(|x|)$ and assume that u differentiable in 0. Show that

$$\nabla u(0) = 0 \,.$$

Exercise 2:
Let $f \in L^1(\mathbb{R}^n)$ and $u(t,x) = \int_{\mathbb{R}^n} \Phi(t, x-y) f(y) dy$. Show that

a) $u \in C^\infty((0,\infty) \times \mathbb{R}^n)$,

b) $u_t = \Delta u$,

c) $\int_{\mathbb{R}^n} u(t,x)dx = \int_{\mathbb{R}^n} f(x)dx$,

d) $\lim_{t\to 0+} \|u(t,\cdot) - f\|_{L^1} = 0$.

Hint: In the lecture you derived the following bound

$$|D^\alpha \Phi(t, x-y)| \leq \widetilde{C}_{\alpha,\delta,x_0} e^{-\frac{|y|^2}{8\delta}} \,,$$

for all $\frac{\delta}{2} \leq t \leq \delta$, $y \in \mathbb{R}^n$, $x \in B_1(x_0)$ and $x_0 \in \mathbb{R}^n$.

Exercise 3:
Consider the following initial boundary value problem on $(0,\infty) \times (0,\infty)$

$$\begin{cases} u_t = u_{xx} \,, & x, t > 0 \\ u_x(t,0) = 0 \,, & \\ u(0,x) = f(x) \,, & f \in L^1((0,\infty)) \end{cases}$$

Determine a solution $u(t,x)$ such that $u(t,\cdot) \in L^1\big((0,\infty)\big)$ for all $t > 0$ and show that

$$\int_0^\infty u(t,x)dx = \int_0^\infty f(x)dx \,.$$