Institute for Analysis
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Classical methods for Partial Differential Equations

07. Problem Sheet

Exercise 1: (Heat equation with Dirichlet boundary conditions)
Consider the following initial boundary value problem on $(0,\infty)\times(0,\infty)$

$$\begin{cases} u_t = u_{xx}\,, & x,t>0 \\ u(t,0)=0\,, & \\ u(0,x)=f(x)\,, & f\in L^1((0,\infty)) \end{cases}$$

Determine a solution $u(t,x)$ such that $u(t,\cdot)\in L^1\big((0,\infty)\big)$ for all $t>0$ and show that

$$\|u(t,x)\|_{L^1} \xrightarrow[t\to\infty]{} 0$$

Exercise 2:
Let $f\in C^{1,2}((0,\infty)\times\mathbb{R}^n)\cap C([0,\infty)\times\mathbb{R}^n)$ be bounded with bounded derivatives. For $\epsilon>0$ and $t\in(0,\infty)$ show the following assertions

a) For $s\in(0,t-\epsilon)$ we have

$$\int_{\mathbb{R}^n}\Delta_x\Phi(t-s,x-y)f(s,y)\,dy = \int_{\mathbb{R}^n}\Phi(t-s,x-y)\Delta_y f(s,y)\,dy\,.$$

b) We have the following equality

$$\begin{aligned}\int_0^{t-\epsilon}\int_{\mathbb{R}^n}\frac{\partial\Phi}{\partial t}(t-s,x-y)f(s,y)\,dy\,ds + \int_{\mathbb{R}^n}\Phi(\epsilon,x-y)f(t-\epsilon,y)\,dy \\ = \int_0^{t-\epsilon}\int_{\mathbb{R}^n}\Phi(t-s,x-y)\frac{\partial f}{\partial s}(s,y)\,dy\,ds + \int_{\mathbb{R}^n}\Phi(t,x-y)f(0,y)\,dy\end{aligned}$$

Exercise 3: (Non-uniquness of Heat equation)
Let a function $g:\mathbb{R}\to\mathbb{R}$ be given by

$$g(t) := \begin{cases}\exp(-t^{-2})\,, & \text{for } t>0, \\ 0\,, & \text{for } t\le 0\end{cases}$$

and $u:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ be defined by

$$u(t,x) := \sum_{k=0}^{\infty}\frac{g^{(k)}(t)}{(2k)!}x^{2k}.$$

— **Please turn the page!** —

Show that $u \in C^{2,1}([0,\infty) \times \mathbb{R}) \cap C([0,\infty) \times \mathbb{R})$ is a solution of the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0\,, & \text{in } [0,\infty) \times \mathbb{R}, \\ u(0,x) = 0\,, & \text{for all } x \in \mathbb{R}. \end{cases}$$

Hint: You are allowed to use the following result:
There exists a $\theta > 0$ such that $\left|g^{(k)}(t)\right| \leq \frac{k!}{(\theta t)^k} \exp\left(-\frac{1}{2}t^{-2}\right)$ for all $t > 0$.