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Classical methods for Partial Differential Equations

09. Problem Sheet

Exercise 1: (Completing the proof of Theorem 5.2)
Let $f \in C^{0,1}([0,\infty) \times \mathbb{R}^n)$, $g \in C^1(\mathbb{R}^n)$ and $b \in \mathbb{R}^n$. In the lecture it was shown that

$$u(t,x) = g(x - bt) + \int_0^t f(s, x - b(t-s))\, ds\,, \tag{1}$$

is (if it exists) the only possible, i.e. unique solution of the transport equation

$$\begin{cases} u_t + b \cdot \nabla_x u = f\,, & (t,x) \in (0,\infty) \times \mathbb{R}^n\,, \\ u(0,x) = g(x)\,, & x \in \mathbb{R}^n\,. \end{cases}$$

Show directly that Eq.(1) is a solution of the transport equation.

Exercise 2: (Existence of the solution given in Theorem 5.3)
Let $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$ and $c > 0$. Show directly that

$$u(t,x) = \frac{g(x+ct) + g(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} h(y)\, dy\,, \quad x \in \mathbb{R}\,,\ t \geq 0\,,$$

is a solution of the homogeneous 1-dim wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx}\,, & x \in \mathbb{R}\,,\ t > 0\,, \\ u(0,x) = g(x)\,, & x \in \mathbb{R}\,, \\ u_t(0,x) = h(x)\,, & x \in \mathbb{R}\,. \end{cases}$$

Exercise 3: (The inhomogeneous 1-dim wave equation)
Let $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$ and $f \in C^1(\overline{\mathbb{R}_+} \times \mathbb{R})$. For $c > 0$, $x \in \mathbb{R}$ and $t \geq 0$ derive a formula for the solution $u(t,x)$ of the inhomogeneous 1-dim wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(t,x)\,, & x \in \mathbb{R}\,,\ t > 0\,, \\ u(0,x) = g(x)\,, & x \in \mathbb{R}\,, \\ u_t(0,x) = h(x)\,, & x \in \mathbb{R}\,. \end{cases}$$

Exercise 4:
Give solutions for the following partial differential equations ($x \in \mathbb{R}$, $t > 0$):

(a) $u_{tt} = a^2\, u_{xx} + \sin(wx)$
$u(0,x) = u_t(0,x) = 0$

(b) $u_{tt} = 4u_{xx} + \sin(x)$
$u(0,x) = \sin(x), u_t(0,x) = 1$

(c) $u_{tt} = a^2\, u_{xx} + \sin(wt)$
$u(0,x) = u_t(0,x) = 0$

(d) $u_t + 5u_x = \sin(x+t)$
$u(0,x) = e^x$

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Exercise 5:
Prove the following property assumed for the derivation of the solution in Example 5.4.
If $g \in C^2([0,\infty))$ with $g(0) = g''(0) = 0$, then $\widetilde{g} \in C^2(\mathbb{R})$, where

$$\widetilde{g}(x) := \begin{cases} g(x)\,, \ x \geq 0\,, \\ -g(-x)\,, \ x < 0\,. \end{cases}$$