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Functional analysis

1. Exercise Sheet

Exercise 1 ((C) Extension principle)

Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be two normed spaces and furthermore assume that Y is complete. Now let $A \subseteq X$ be a dense subset of X and $f\colon A \to Y$ a function. Show that, if the function f is uniformly continuous, then there is an unique continuous function $\widetilde{f}\colon X \to Y$ such that $\widetilde{f}|_A = f$ holds. Proof with a counterexample the necessity of the uniform continuity.

Exercise 2 (Metric spaces)

1. Let (X, d_X) be a metric space.

   (M1) Show: If $h\colon X \to X$ is an injective map, then $d_h(x,y) := d_X(h(x), h(y))$ is a metric on X.

   (M2) Now we set $X := \mathbb{R}$ with the standard metric (=absolute value). Is the metric space $(\mathbb{R}, d_{\arctan})$ complete?

2. Proof: We have via

$$\|u\|_{L^2([-1,1])} := \left(\int_{-1}^{1} |u(x)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}$$

   a norm on $C^0([-1,1],\mathbb{K})$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, but the space $\left(C^0([-1,1],\mathbb{K}), \|\cdot\|_{L^2([-1,1])}\right)$ is not a Banach space.

Exercise 3 ((C) Banach spaces)

Let $(X, \|\cdot\|_X)$ be a normed space. Show the following:

(C1) X is a Banach space if and only if for all sequences $(x_n)_{n\in\mathbb{N}} \subseteq X$ with $\sum\limits_{n=1}^{\infty} \|x_n\|_X < \infty$ there exists an element $x \in X$ with

$$\lim_{N\to\infty} \left\| x - \sum_{n=1}^{N} x_n \right\|_X = 0.$$

(C2) We define the sequence-space l^p for $p \in [1,\infty]$ via

$$l^p := \left\{ x = (x_n)_{n\in\mathbb{N}} \subseteq \mathbb{K}\colon \; \|x\|_{l^p} := \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} < \infty \right\} \text{ if } p < \infty,$$

$$l^\infty := \left\{ x = (x_n)_{n\in\mathbb{N}} \subseteq \mathbb{K}\colon \; \|x\|_{l^\infty} := \sup_{n\in\mathbb{N}} |x_n| < \infty \right\}$$

for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Show that for every $p \in [1,\infty]$ the space $(l^p, \|\cdot\|_{l^p})$ is a Banach space.

Exercise 4 ()

Let (X, d_X) be a metric space and let $Y \subseteq X$ be a subset of X. Show the following:

(D1) If (Y, d_X) is complete, then Y is closed in X.

(D2) If (X, d_X) is complete and Y is closed in X, then (Y, d_X) is also complete.

Exercise Sheets

There will be one exercise sheet every week with three or four problems which you can find on the webpage. You can hand in your solution of the two exercises which are labeled with a 'C' and then the tutor will correct it. Please deliver your solution in the box on the ground floor of the math building or in the problem class. In the problem class we will discuss the solutions. If I have not enough time, I will upload my sketches to the missing problems on the webpage.