Functional analysis

2. Exercise Sheet - Solutions

Exercise 1  (Cartesian Product of Banach spaces)
Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and we define the map

$$\|\cdot\|_{X \times Y} : X \times Y \rightarrow \mathbb{R}, \ (x,y) \mapsto \max \{\|x\|_X,\|y\|_Y\}.$$ 

1. Show that $(X \times Y, \|\cdot\|_{X \times Y})$ is a normed space.
2. Show that if $X$ and $Y$ are Banach spaces, then $(X \times Y, \|\cdot\|_{X \times Y})$ is also a Banach space.

Exercise 2  ((C) Multiplication Operators)
1. Let $\Omega \subseteq \mathbb{R}^d, X := C_0^0(\Omega, \mathbb{K}) := \{f : \Omega \rightarrow \mathbb{K} : f \text{ is bounded and continuous on } \Omega\}$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}, m \in X$ and $M$ be the Multiplication operator given by

$$M : X \rightarrow X, \ Mf = m \cdot f.$$ 

Show that $M$ is well-defined with operator norm $\|M\| = \|m\|_{C_0^0(\Omega, \mathbb{K})}$.
2. Let $(\Omega, \mathcal{A}, \mu)$ a $\sigma$-finite measure space and $m : \Omega \rightarrow \mathbb{K}$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be $\mu$-measurable, $p \in [1, \infty]$. We define formally the multiplication operator $M$ by

$$Mf = m \cdot f.$$ 

Show the following:

$$M \in B(L^p(\Omega, \mu), L^p(\Omega, \mu)) \iff m \in L^\infty(\Omega, \mu).$$

In this case we have for the operator norm of $M$:

$$\|M\| = \|m\|_{L^\infty(\Omega, \mu)}.$$ 

Exercise 3  ((C) Integral operator)
Set $I := [0, 1] \subseteq \mathbb{R}$ and let $k \in C^0(I \times I, \mathbb{K})$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be a continuous function on $I \times I$. We define the integral operator $T : (C_0^0(I, \mathbb{K}), \|\cdot\|_{C_0^0(I,K)}) \rightarrow (C^0(I, \mathbb{K}), \|\cdot\|_{C^0(I,K)})$ with kernel $k$ by

$$Tf(x) = \int_0^1 k(x,y)f(y)dy, \ x \in I.$$ 

Show that the integral operator $T$ is well-defined and compute the operator norm $\|T\|$.

Exercise 4  (Finite dimensional subspaces)
Let $(X, \|\cdot\|_X)$ be a normed space and $U \subseteq X$ be a finite dimensional subspace. Show that $U$ is closed in $X$.

Solution of Exercise 4
We denote $d := \dim(U) < \infty$ and let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be such that $X$ is a $\mathbb{K}$-vector space. We choose $e_1, \ldots, e_d \in U$ such that $\|e_i\|_X = 1$ for $i = 1, \ldots, d$ and span $\{e_1, \ldots, e_d\} = U$. Let $(x_n)_{n \in \mathbb{N}} \subseteq U$ be a converging sequence in $X$, i.e. there is an element $x \in X$ such that for every $\varepsilon > 0$ we find an index $n_0(\varepsilon) =: n_0 \in \mathbb{N}$ with

$$\|x_n - x\|_X < \frac{\varepsilon}{2} \text{ for all } n \in \mathbb{N} \text{ with } n \geq n_0.$$
By assumption there is for every index $i \in \{1, \ldots, d\}$ a sequence $\left( \lambda_i^{(n)} \right)_{n \in \mathbb{N}} \subseteq \mathbb{K}$ such that
\[ x_n = \sum_{i=1}^{d} \lambda_i^{(n)} e_i \text{ for all } n \in \mathbb{N}. \]

By the lecture we know that $\| \cdot \|_{\mathbb{X}}$ and $\| \cdot \|_1$ are equivalent on $U$, i.e. there is a constant $C \geq 1$ such that
\[ \frac{1}{C} \| y \|_1 \leq \| y \|_{\mathbb{X}} \leq C \| y \|_1 := \sum_{i=1}^{d} \lambda_i(y) \text{ for all } y = \sum_{i=1}^{d} \lambda_i(y) e_i \in U. \]

Especially the sequence $(x_n)_{n \in \mathbb{N}}$ is a Cauchy-sequence, so choose $n_1 \in \mathbb{N}$ with $n_1 \geq n_0$ such that
\[ \| x_n - x_m \|_{\mathbb{X}} < \frac{\varepsilon}{C} \text{ for all } n, m \in \mathbb{N} \text{ with } n, m \geq n_1. \]

This implies that for every $i_0 \in \{1, \ldots, d\}$ and for $n, m \in \mathbb{N}$ with $n, m \geq n_1$ we have
\[ \left| \lambda_{i_0}^{(n)} - \lambda_{i_0}^{(m)} \right| \leq \sum_{i=1}^{d} \left| \lambda_i^{(n)} - \lambda_i^{(m)} \right| \leq C \| x_n - x_m \|_{\mathbb{X}} < C \cdot \frac{\varepsilon}{C} = \varepsilon, \]

i.e. $\left( \lambda_i^{(n)} \right)_{n \in \mathbb{N}} \subseteq \mathbb{K}$ is a Cauchy-sequence in $\mathbb{K}$ for all $i \in \{1, \ldots, d\}$. Since $(\mathbb{K}, | \cdot |)$ is complete the Cauchy-sequence $\left( \lambda_i^{(n)} \right)_{n \in \mathbb{N}}$ converge in $\mathbb{K}$ to an element $\lambda_i \in \mathbb{K}$ for every $i \in \{1, \ldots, d\}$. Set $x^* := \sum_{i=1}^{d} \lambda_i e_i \in U$. Now choose an index $n_2 \in \mathbb{N}$ with $n_2 \geq n_1$ and such that
\[ \| x^* - x_{n_2} \|_1 < \frac{\varepsilon}{2C} \text{ for all } n \in \mathbb{N} \text{ with } n \geq n_2. \]

Then we conclude:
\[ \| x^* - x \|_{\mathbb{X}} \leq \| x^* - x_{n_2} \|_{\mathbb{X}} + \| x_{n_2} - x \|_{\mathbb{X}} \leq C \| x^* - x_{n_2} \|_1 + \| x_{n_2} - x \|_{\mathbb{X}} < C \cdot \frac{\varepsilon}{2C} + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

Since $\varepsilon > 0$ was arbitrary: $x = x^* \in U$, i.e. $U$ is closed in $\mathbb{X}$.

\[ \square \]