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Functional analysis

2. Exercise Sheet - Solutions

Exercise 1 (Cartesian Product of Banach spaces)

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and we define the map

$$\|\cdot\|_{X\times Y} : X \times Y \to \mathbb{R},\ (x,y) \mapsto \max\{\|x\|_X, \|y\|_Y\}.$$

1. Show that $(X \times Y, \|\cdot\|_{X\times Y})$ is a normed space.
2. Show that if X and Y are Banach spaces, then $(X \times Y, \|\cdot\|_{X\times Y})$ is also a Banach space.

Exercise 2 ((C) Multiplication Operators)

1. Let $\Omega \subseteq \mathbb{R}^d$, $X := C_b^0(\Omega, \mathbb{K}) := \{f : \Omega \to \mathbb{K} : f \text{ is bounded and continuous on } \Omega\}$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $m \in X$ and M be the Multiplication operator given by

$$M : X \to X,\ Mf = m \cdot f.$$

Show that M is well-defined with operator norm $\|M\| = \|m\|_{C^0(\Omega,\mathbb{K})}$.
2. Let $(\Omega, \mathcal{A}, \mu)$ a σ-finite measure space and $m : \Omega \to \mathbb{K}$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be μ-measurable, $p \in [1, \infty]$. We define formally the multiplication operator M by

$$Mf = m \cdot f.$$

Show the following:

$$M \in B\left(L^p(\Omega,\mu), L^p(\Omega,\mu)\right) \ \Leftrightarrow\ m \in L^\infty(\Omega,\mu).$$

In this case we have for the operator norm of M:

$$\|M\| = \|m\|_{L^\infty(\Omega,\mu)}.$$

Exercise 3 ((C) Integral operator)

Set $I := [0,1] \subseteq \mathbb{R}$ and let $k \in C^0(I \times I, \mathbb{K})$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be a continuous function on $I \times I$. We define the integral operator $T : \left(C^0(I,\mathbb{K}), \|\cdot\|_{C^0(I,\mathbb{K})}\right) \to \left(C^0(I,\mathbb{K}), \|\cdot\|_{C^0(I,\mathbb{K})}\right)$ with kernel k by

$$Tf(x) = \int_0^1 k(x,y) f(y)\,\mathrm{d}y,\ x \in I.$$

Show that the integral operator T is well-defined and compute the operator norm $\|T\|$.

Exercise 4 (Finite dimensional subspaces)

Let $(X, \|\cdot\|_X)$ be a normed space and $U \subseteq X$ be a finite dimensional subpace. Show that U is closed in X.

Solution of Exercise 4

We denote $d := \dim(U) < \infty$ and let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be such that X is a $\mathbb{K}$-vector space. We choose $e_1, \ldots, e_d \in U$ such that $\|e_i\|_X = 1$ for $i = 1, \ldots, d$ and $\operatorname{span}\{e_1, \ldots, e_d\} = U$. Let $(x_n)_{n\in\mathbb{N}} \subseteq U$ be a converging sequence in X, i.e. there is an element $x \in X$ such that for every $\varepsilon > 0$ we find an index $n_0(\varepsilon) =: n_0 \in \mathbb{N}$ with

$$\|x_n - x\|_X < \frac{\varepsilon}{2} \text{ for all } n \in \mathbb{N} \text{ with } n \geq n_0.$$

By assumption there is for every index $i \in \{1, \dots, d\}$ a sequence $\left(\lambda_i^{(n)}\right)_{n\in\mathbb{N}} \subseteq \mathbb{K}$ such that

$$x_n = \sum_{i=1}^{d} \lambda_i^{(n)} e_i \text{ for all } n \in \mathbb{N}.$$

By the lecture we know that $\|\cdot\|_X$ and $\|\cdot\|_1$ are equivalent on U, i.e. there is a constant $C \geq 1$ such that

$$\frac{1}{C} \|y\|_1 \leq \|y\|_X \leq C \|y\|_1 := \sum_{i=1}^{d} \lambda_i(y) \text{ for all } y = \sum_{i=1}^{d} \lambda_i(y) e_i \in U.$$

Especially the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence, so choose $n_1 \in \mathbb{N}$ with $n_1 \geq n_0$ such that

$$\|x_n - x_m\|_X < \frac{\varepsilon}{C} \text{ for all } n, m \in \mathbb{N} \text{ with } n, m \geq n_1.$$

This implies that for every $i_0 \in \{1, \dots, d\}$ and for $n, m \in \mathbb{N}$ with $n, m \geq n_1$ we have

$$\left|\lambda_{i_0}^{(n)} - \lambda_{i_0}^{(m)}\right| \leq \sum_{i=1}^{d} \left|\lambda_i^{(n)} - \lambda_i^{(m)}\right| \leq C \|x_n - x_m\|_X < C \cdot \frac{\varepsilon}{C} = \varepsilon,$$

i.e. $\left(\lambda_i^{(n)}\right)_{n\in\mathbb{N}} \subseteq \mathbb{K}$ is a Cauchy-sequence in $\mathbb{K}$ for all $i \in \{1, \dots, d\}$. Since $(\mathbb{K}, |\cdot|)$ is complete the Cauchy-sequence $\left(\lambda_i^{(n)}\right)_{n\in\mathbb{N}}$ converge in $\mathbb{K}$ to an element $\lambda_i \in \mathbb{K}$ for every $i \in \{1, \dots, d\}$. Set $x^* := \sum_{i=1}^{d} \lambda_i e_i \in U$. Now choose an index $n_2 \in \mathbb{N}$ with $n_2 \geq n_1$ and such that

$$\|x^* - x_n\|_1 < \frac{\varepsilon}{2C} \text{ for all } n \in \mathbb{N} \text{ with } n \geq n_2.$$

Then we conclude:

$$\|x^* - x\|_X \leq \|x^* - x_{n_2}\|_X + \|x_{n_2} - x\|_X \leq C \|x^* - x_{n_2}\|_1 + \|x_{n_2} - x\|_X < C \cdot \frac{\varepsilon}{2C} + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary: $x = x^* \in U$, i.e. U is closed in X. □