Functional analysis

3. Exercise Sheet

Exercise 1  (Hausdorff metric)

We denote with \( \mathcal{A} \) the set of all nonempty closed and bounded subsets of \( \mathbb{R}^d \).

1. Show that the map
\[
d_H(A_1, A_2) := \inf\{\rho > 0 : A_1 \subseteq B_\rho(A_2) \text{ and } A_2 \subseteq B_\rho(A_1)\}
\]
for \( A_1, A_2 \in \mathcal{A} \), where \( B_\rho(A) := \{x \in \mathbb{R}^d : \text{dist} (x,A) < \rho\} \) for nonempty \( A \subseteq \mathbb{R}^d \), is a metric on \( \mathcal{A} \), i.e. \( (\mathcal{A},d_H) \) is a metric space.

2. Show that the set
\[
K := \{A \in \mathcal{A} : A \subseteq B(R(0))\}
\]
is compact in \( (\mathcal{A},d_H) \).

Exercise 2  (The Ehrling Lemma)

Let \( X \) be a normed space with three norms \( \|\cdot\|_a \), \( \|\cdot\|_b \) and \( \|\cdot\|_c \). These norms have the following two properties:

1. For every sequence in \( X \) which is bounded in the \( \|\cdot\|_a \)-norm there is a subsequence which converges with respect to \( \|\cdot\|_b \)-norm.
2. There is a constant \( \Lambda > 0 \) such that
\[
\|x\|_c \leq \Lambda \|x\|_b
\]
for all \( x \in X \).

Show that for every \( \varepsilon > 0 \) there is a constant \( C_\varepsilon > 0 \) such that
\[
\|x\|_b \leq \varepsilon \|x\|_a + C_\varepsilon \|x\|_c
\]
for all \( x \in X \).

Exercise 3  (Precompactness in \( C^0(I,\mathbb{K}) \))

Let \( I := [0,1] \) be the unit interval in \( \mathbb{R} \). Which of the following families is precompact in \( C^0(I,\mathbb{K}) \) with \( \mathbb{K} \in \{\mathbb{R},\mathbb{C}\} \) with respect to the sup-norm?

1. \( A := \{f_n : n \in \mathbb{N}\} \) with \( f_n(x) = \sin(x+n) \) for \( x \in I \) and \( n \in \mathbb{N} \).
2. \( B := \{f_n : n \in \mathbb{N}\} \) with \( f_n(x) = \sin(nx) \) for \( x \in I \) and \( n \in \mathbb{N} \).

Exercise 4  (Relativ compactness)

Set \( I := [0,1] \subseteq \mathbb{R} \) and let \( k \in C^0(I \times I,\mathbb{K}) \) with \( \mathbb{K} \in \{\mathbb{R},\mathbb{C}\} \) be a continuous function on \( I \times I \). We define the integral operator \( T : \left(C^0(I,\mathbb{K}),\|\cdot\|_{C^0(I,\mathbb{K})}\right) \to \left(C^0(I,\mathbb{K}),\|\cdot\|_{C^0(I,\mathbb{K})}\right) \) with kernel \( k \) by
\[
Tf(x) = \int_0^1 k(x,y)f(y)dy, \quad x \in I.
\]

On Exercise sheet 2, Exercise 3 you saw that \( T \) is a well-defined linear and continuous/bounded operator on \( X := C^0(I,\mathbb{K}) \). Show that the set
\[
K := \{Tf : f \in B_X^r(0)\}
\]
is relativ compact in \( X \), i.e. the closure of \( K \) is compact in \( X \), where \( B_X^r(g) \) is the ball in \( X \) with radius \( r > 0 \) and center \( g \in X \).