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Prof. Dr. Tobias Lamm
M.Sc. Michael Ullmann

Functional analysis

3. Exercise Sheet

Exercise 1 (Hausdorff metric)

We denote with $\mathcal{A}$ the set of all nonempty closed and bounded subsets of $\mathbb{R}^d$.

1. Show that the map d_H

$$d_H(A_1, A_2) := \inf\{\rho > 0\colon A_1 \subseteq B_\rho(A_2) \text{ and } A_2 \subseteq B_\rho(A_1)\} \text{ for } A_1, A_2 \in \mathcal{A},$$

where $B_\rho(A) := \left\{x \in \mathbb{R}^d\colon \operatorname{dist}(x, A) < \rho\right\}$ for nonempty $A \subseteq \mathbb{R}^d$, is a metric on $\mathcal{A}$, i.e. $(\mathcal{A}, d_H)$ is a metric space.

2. Show that the set

$$K := \left\{A \in \mathcal{A}\colon A \subseteq \overline{B_R(0)}\right\}$$

is compact in $(\mathcal{A}, d_H)$.

Exercise 2 (The Ehrling Lemma)

Let X be a normed space with three norms $\|\cdot\|_a, \|\cdot\|_b$ and $\|\cdot\|_c$. These norms have the following two properties:

(1) For every sequence in X which is bounded in the $\|\cdot\|_a$-norm there is a subsequence which converges with respect to $\|\cdot\|_b$-norm.

(2) There is a constant $\Lambda > 0$ such that

$$\|x\|_c \leq \Lambda \|x\|_b \text{ for all } x \in X.$$

Show that for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$\|x\|_b \leq \varepsilon \|x\|_a + C_\varepsilon \|x\|_c \text{ for all } x \in X.$$

Exercise 3 (Precompactness in $C^0(I, \mathbb{K})$)

Let $I := [0, 1]$ be the unit interval in $\mathbb{R}$. Which of the following families is precompact in $C^0(I, \mathbb{K})$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with respect to the sup-norm?

(1) $A := \{f_n\colon n \in \mathbb{N}\}$ with $f_n(x) = \sin(x + n)$ for $x \in I$ and $n \in \mathbb{N}$.

(2) $B := \{f_n\colon n \in \mathbb{N}\}$ with $f_n(x) = \sin(nx)$ for $x \in I$ and $n \in \mathbb{N}$.

Exercise 4 (Relativ compactness)

Set $I := [0, 1] \subseteq \mathbb{R}$ and let $k \in C^0(I \times I, \mathbb{K})$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be a continuous function on $I \times I$. We define the integral operator $T\colon \left(C^0(I, \mathbb{K}), \|\cdot\|_{C^0(I,\mathbb{K})}\right) \to \left(C^0(I, \mathbb{K}), \|\cdot\|_{C^0(I,\mathbb{K})}\right)$ with kernel k by

$$Tf(x) = \int_0^1 k(x, y) f(y) \mathrm{d}y, \; x \in I.$$

On Exercise sheet 2, Exercise 3 you saw that T is a well-defined linear and continuous/ bounded operator on $X := C^0(I, \mathbb{K})$. Show that the set

$$K := \left\{Tf\colon f \in \overline{B_1^X(0)}\right\}$$

is relativ compact in X, i.e. the closure of K is compact in X, where $B_r^X(g)$ is the ball in X with radius $r > 0$ and center $g \in X$.