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Functional analysis

Solutions to 3. Exercise Sheet

Exercise 1 (Hausdorff metric)

We denote with $\mathcal{A}$ the set of all nonempty closed and bounded subsets of $\mathbb{R}^d$.
1. Show that the map d_H

$$d_H(A_1, A_2) := \inf\{\rho > 0 \colon A_1 \subseteq B_\rho(A_2) \text{ and } A_2 \subseteq B_\rho(A_1)\} \text{ for } A_1, A_2 \in \mathcal{A},$$

where $B_\rho(A) := \left\{x \in \mathbb{R}^d \colon \operatorname{dist}(x, A) < \rho\right\}$ for nonempty $A \subseteq \mathbb{R}^d$, is a metric on $\mathcal{A}$, i.e. $(\mathcal{A}, d_H)$ is a metric space.
2. Show that the set

$$K := \left\{A \in \mathcal{A} \colon A \subseteq \overline{B_R(0)}\right\}$$

is compact in $(\mathcal{A}, d_H)$.

Solution of Exercise 1

2. We can prove easily a much more general result if we denote for a nonempty set $M \subseteq \mathbb{R}^d$ the set $K(M)$ by

$$K(M) := \{A \in \mathcal{A} \colon A \subseteq M\}.$$

We split our proof in two lemmas:
Lemma 1. (Extension Lemma) Let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be a Cauchy-sequence in $K(M)$ and let $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ be a strictly monotone increasing sequence. If $(x_{n_k})_{k\in\mathbb{N}} \subseteq \mathbb{R}^d$ is a Cauchy-sequence with $x_{n_k} \in A_{n_k}$ for all $k \in \mathbb{N}$, then there is a Cauchy-sequence $(y_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^d$ with $y_n \in A_n$ for all $n \in \mathbb{N}$ and $y_{n_k} = x_{n_k}$ for all $k \in \mathbb{N}$.
Lemma 2. If the set M is in addition totally bounded/ complete/ compact the set $K(M)$ is also totally bounded/ complete/ compact.
Proof: (1) Let M be totally bounded and let ε be positive. Then we find finite many elements $x_1, \ldots, x_n \in M$ with

$$\min_{i=1,\ldots,n} |x - x_i| < \varepsilon \text{ for all } x \in M.$$

Let $A \in K(M)$ be arbitrary and set $S_A := \{x_i \colon \operatorname{dist}(x_i, A) < \varepsilon\}$, then we have

$$d_H(S_A, A) < \varepsilon.$$

This implies

$$K(M) \subseteq \bigcup_{i=1}^{n} B_\varepsilon(\{x_i\}),$$

i.e. $K(M)$ is totally bounded.
(2) Let M be complete and $(A_n)_{n\in\mathbb{N}} \subseteq K(M)$ be a Cauchy-sequence, $\varepsilon > 0$. With the Cauchy-property we can assume w.l.o.g that

$$d_H(A_n, A_{n+1}) < 2^{-n} \text{ for all } n \in \mathbb{N},$$

since otherwise we choose such an subsequence. We have for $n \in \mathbb{N}$:

$$A_n \subseteq B_{2^{-n}}(A_{n+1}) \text{ and } A_{n+1} \subseteq B_{2^{-n}}(A_n).$$

Then we get for every $N \in \mathbb{N}$ a sequence $(x_n)_{n\geq N}$ with $x_n \in A_n$ and $|x_n - x_{n+1}| < 2^{-n}$ for all $n \in \mathbb{N}$ with $n \geq N$. We calculate by triangle-inequality for $n, m \in \mathbb{N}$ with $N \leq n \leq m$:

$$|x_n - x_m| \leq \sum_{j=0}^{m-1} |x_{n+j} - x_{n+j+1}|$$

$$
\begin{aligned}
&< \sum_{j=0}^{m-1} 2^{-(n+j)} = 2^{-n} \sum_{j=0}^{m-1} 2^{-j} \\
&< 2^{-n} \sum_{j=0}^{\infty} 2^{-j} = 2^{-n+1} \to 0
\end{aligned}
$$

as $m, n \to \infty$, i.e. $(x_n)_{n\geq N}$ is a Cauchy-sequence in M. Since M is complete there is an element $x \in M$ such that $\lim_{n\to\infty} x_n = x$ in M. Then we have the estimate

$$
\begin{aligned}
|x_n - x| &= \lim_{m\to\infty,\ m\geq n} |x_n - x_m| \\
&\leq 2^{-n+1}.
\end{aligned}
$$

We set as A the set of all such Limites in M, then by construction the set A is nonempty. Let $x \in A$ then there is a sequence $(x_n)_{n\in\mathbb{N}} \subseteq M$ with $x_n \in A_n$ and $|x_n - x| \leq 2^{-n+1}$ for every $n \in \mathbb{N}$, i.e.

$$
A \subseteq B_{2^{-n+1}}(A_n) \text{ for every } n \in \mathbb{N},
$$

i.e. A is bounded, and so $\overline{A} \in \mathcal{A}$. Let $\varepsilon > 0$ be arbitrary, then we find a natural number $N \in \mathbb{N}$ such that

$$
\varepsilon > 2^{-N}.
$$

Let $n \in \mathbb{N}$ be with $n \geq N + 1$. For every $x_n \in A_n$ we find an element $x \in \overline{A}$ with

$$
|x_n - x| \leq 2^{-n+1} \leq 2^{-(N+1)+1} = 2^{-N} < \varepsilon,
$$

i.e.

$$
A_n \subseteq B_{2^{-n+1}}\left(\overline{A}\right).
$$

This implies

$$
d_H\left(A_n, \overline{A}\right) < 2^{-n+1} < \varepsilon \text{ for all } n \in \mathbb{N} \text{ with } n \geq N + 1,
$$

i.e. $(A_n)_{n\in\mathbb{N}}$ converge to $\overline{A}$.
Now it's left to show that the set A is closed under above construction. Let $a \in \overline{A}$ be an arbitrary limit point and choose $(a_k)_{k\in\mathbb{N}} \subseteq A$ such that $\lim_{k\to\infty} a_k = a$ in M. Then by construction we find for every $k \in \mathbb{N}$ a sequence $\left(y_n^{(k)}\right)_{n\in\mathbb{N}}$ with $y_n^{(k)} \in A_n$ and $\lim_{n\to\infty} y_n^{(k)} = a_k$. Now choose $n_1 \in \mathbb{N}$ such that $\left|y_{n_1}^{(1)} - a_1\right| < 1$. Do it again choose $n_2 \in \mathbb{N}$ with $n_2 > n_1$ with $|y_{n_2} - a_2| < \frac{1}{2}$. Repeat this prozess, then we get a strictly monotone increasing sequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ with $y_{n_k}^{(k)} \in A_{n_k}$ and $|y_{n_k} - a_k| < \frac{1}{k}$ for all $k \in \mathbb{N}$. By triangle-inequality it follows:

$$
\left|y_{n_k}^{(k)} - a\right| \leq \left|y_{n_k}^{(k)} - a_k\right| + |a_k - a| < \frac{1}{k} + \varepsilon
$$

for $k \in \mathbb{N}$ large enough, i.e.

$$
\lim_{k\to\infty} y_{n_k}^{(k)} = a.
$$

Since every convergent sequence is a Cauchy-sequence, the Extension Lemma gives us a Cauchy-sequence $(x_n)_{n\in\mathbb{N}} \subseteq M$ with $x_n \in A_n$ and $x_{n_k} = y_{n_k}^{(k)}$ for every $n, k \in \mathbb{N}$. This conludes $\lim_{n\to\infty} x_n = a$ and that means $a \in A$, i.e. A is closed, since a was arbitrary. Everything together we see that $K(M)$ is complete.
(3) Let M be compact, then we know by definition that M is totally bounded and complete. This implies with (1) and (2) that $K(M)$ is also totally bounded and complete. By definition of compactness again we know that $K(M)$ is compact. □

Exercise 2 (The Ehrling Lemma)

Let X be a normed space with three norms $\|\cdot\|_a, \|\cdot\|_b$ and $\|\cdot\|_c$. These norms have the following two properties:

(1) For every sequence in X which is bounded in the $\|\cdot\|_a$-norm there is a subsequence which converges with respect to $\|\cdot\|_b$-norm.

(2) There is a constant $\Lambda > 0$ such that

$$
\|x\|_c \leq \Lambda \|x\|_b \text{ for all } x \in X.
$$

Show that for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$\|x\|_b \leq \varepsilon \|x\|_a + C_\varepsilon \|x\|_c \text{ for all } x \in X.$$

Exercise 3 (Precompactness in $C^0(I, \mathbb{K})$)

Let $I := [0, 1]$ be the unit interval in $\mathbb{R}$. Which of the following families is precompact in $C^0(I, \mathbb{K})$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with respect to the sup-norm?

(1) $A := \{f_n : n \in \mathbb{N}\}$ with $f_n(x) = \sin(x + n)$ for $x \in I$ and $n \in \mathbb{N}$.

(2) $B := \{f_n : n \in \mathbb{N}\}$ with $f_n(x) = \sin(nx)$ for $x \in I$ and $n \in \mathbb{N}$.

Exercise 4 (Relativ compactness)

Set $I := [0, 1] \subseteq \mathbb{R}$ and let $k \in C^0(I \times I, \mathbb{K})$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be a continuous function on $I \times I$. We define the integral operator $T\colon \left(C^0(I, \mathbb{K}), \|\cdot\|_{C^0(I,\mathbb{K})}\right) \to \left(C^0(I, \mathbb{K}), \|\cdot\|_{C^0(I,\mathbb{K})}\right)$ with kernel k by

$$Tf(x) = \int_0^1 k(x, y)f(y)\mathrm{d}y, \ x \in I.$$

On Exercise sheet 2, Exercise 3 you saw that T is a well-defined linear and continuous/ bounded operator on $X := C^0(I, \mathbb{K})$. Show that the set

$$K := \left\{Tf : f \in \overline{B_1^X(0)}\right\}$$

is relativ compact in X, i.e. the closure of K is compact in X, where $B_r^X(g)$ is the ball in X with radius $r > 0$ and center $g \in X$.

Solution of Exercise 4

We want to use the theorem of Arzela-Ascoli. The interval I is obviously compact in $\mathbb{R}$, since it's closed and bounded (Heine-Borel). The family K is pointwise bounded, since for any $x \in I$, $g \in K$ we find a function $f \in \overline{B_1^X(0)}$ with $Tf = g$ and then we get

$$\begin{aligned}
|Tf(x)| &= \left|\int_0^1 k(x, y)f(y)\mathrm{d}y\right| \leq \int_0^1 |k(x, y)| \cdot |f(y)|\mathrm{d}y \\
&\leq \int_0^1 \|k\|_{C^0(I\times I,\mathbb{K})} \cdot \|f\|_{C^0(I,\mathbb{K})} \,\mathrm{d}y = \|k\|_{C^0(I\times I,\mathbb{K})} \cdot \|f\|_{C^0(I,\mathbb{K})} \\
&\leq \|k\|_{C^0(I\times I,\mathbb{K})}
\end{aligned}$$

by triangle-inequality. The family K is uniformly equicontinuous, since for every $\varepsilon > 0$ we find with the uniform continuity of k on $I \times I$ a $\delta(\varepsilon) =: \delta > 0$ such that for pairs $(x, y), (x', y') \in I \times I$ with $|(x, y) - (x', y')| < \delta$ we have

$$|k(x, y) - k(x', y')| < \varepsilon.$$

Then we get for $g \in K$ and $f \in \overline{B_1^X(0)}$ with $Tf = g$:

$$\begin{aligned}
|Tf(x) - Tf(x')| &= \left|\int_0^1 (k(x, y) - k(x', y)) f(y)\mathrm{d}y\right| \\
&\leq \int_0^1 |k(x, y) - k(x', y)| \cdot |f(y)|\mathrm{d}y \\
&\leq \int_0^1 |k(x, y) - k(x', y)| \,\mathrm{d}y \,\|f\|_{C^0(I,\mathbb{K})} \\
&< \int_0^1 \varepsilon \mathrm{d}y \,\|f\|_{C^0(I,\mathbb{K})} \\
&\leq \varepsilon
\end{aligned}$$

for all $x, x' \in I$ with $|x - x'| < \delta$, i.e. the family K is uniformly equicontinuous. The theorem of Arzela-Ascoli implies now, that the set K is precompact in $C^0(I, \mathbb{K})$ with respect to the sup-norm. □