Functional analysis

Solutions to 3. Exercise Sheet

Exercise 1  (Hausdorff metric)

We denote with \( \mathcal{A} \) the set of all nonempty closed and bounded subsets of \( \mathbb{R}^d \).

1. Show that the map \( d_H(A_1, A_2) := \inf \{ \rho > 0 : A_1 \subseteq B_\rho(A_2) \text{ and } A_2 \subseteq B_\rho(A_1) \} \) for \( A_1, A_2 \in \mathcal{A} \), where \( B_\rho(A) := \{ x \in \mathbb{R}^d : \text{dist}(x, A) < \rho \} \) for nonempty \( A \subseteq \mathbb{R}^d \), is a metric on \( \mathcal{A} \), i.e. \( (\mathcal{A}, d_H) \) is a metric space.

2. Show that the set \( K := \{ A \in \mathcal{A} : A \subseteq B_\rho(0) \} \) is compact in \( (\mathcal{A}, d_H) \).

Solution of Exercise 1

2. We can prove easily a much more general result if we denote for a nonempty set \( M \subseteq \mathbb{R}^d \) the set \( K(M) := \{ A \in \mathcal{A} : A \subseteq M \} \).

We split our proof in two lemmas:

Lemma 1. (Extension Lemma) Let \( (A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A} \) be a Cauchy-sequence in \( K(M) \) and let \( (n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N} \) be a strictly monotone increasing sequence. If \( (x_{n_k})_{k \in \mathbb{N}} \subseteq \mathbb{R}^d \) is a Cauchy-sequence with \( x_{n_k} \in A_{n_k} \) for all \( k \in \mathbb{N} \), then there is a Cauchy-sequence \( (y_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}^d \) with \( y_n \in A_n \) for all \( n \in \mathbb{N} \) and \( y_{n_k} = x_{n_k} \) for all \( k \in \mathbb{N} \).

Lemma 2. If the set \( M \) is in addition totally bounded/ complete/ compact the set \( K(M) \) is also totally bounded/ complete/ compact.

Proof: (1) Let \( M \) be totally bounded and let \( \varepsilon \) be positive. Then we find finite many elements \( x_1, \ldots, x_n \in M \) with

\[
\min_{i=1, \ldots, n} |x_i| < \varepsilon \text{ for all } x \in M.
\]

Let \( A \in K(M) \) be arbitrary and set \( S_A := \{ x_i : \text{dist}(x_i, A) < \varepsilon \} \), then we have

\[
d_H(S_A, A) < \varepsilon.
\]

This implies

\[
K(M) \subseteq \bigcup_{i=1}^{n} B_\varepsilon([x_i]),
\]

i.e. \( K(M) \) is totally bounded.

(2) Let \( M \) be complete and \( (A_n)_{n \in \mathbb{N}} \subseteq K(M) \) be a Cauchy-sequence, \( \varepsilon > 0 \). With the Cauchy-property we can assume w.l.o.g that

\[
d_H(A_n, A_{n+1}) < 2^{-n} \text{ for all } n \in \mathbb{N},
\]

since otherwise we choose such an subsequence. We have for \( n \in \mathbb{N} \):

\[
A_n \subseteq B_{2^{-n}}(A_{n+1}) \text{ and } A_{n+1} \subseteq B_{2^{-n}}(A_n).
\]

Then we get for every \( N \in \mathbb{N} \) a sequence \( (x_n)_{n \geq N} \) with \( x_n \in A_n \) and \( |x_n - x_{n+1}| < 2^{-n} \) for all \( n \in \mathbb{N} \) with \( n \geq N \). We calculate by triangle-inequality for \( n, m \in \mathbb{N} \) with \( N \leq n \leq m \):

\[
|x_n - x_m| \leq \sum_{j=0}^{m-1} |x_{n+j} - x_{n+j+1}|
\]
This implies \( i.e. \ (x_n)_{n \geq N} \) is a Cauchy-sequence in \( M \). Since \( M \) is complete there is an element \( x \in M \) such that \( \lim_{n \to \infty} x_n = x \) in \( M \). Then we have the estimate

\[
|x_n - x| = \lim_{m \to \infty, \ m \geq n} |x_n - x_m| \\
\leq 2^{-n+1}.
\]

We set as \( A \) the set of all such Limites in \( M \), then by construction the set \( A \) is nonempty. Let \( x \in A \) then there is a sequence \( (x_n)_{n \in \mathbb{N}} \subseteq M \) with \( x_n \in A \) and \( |x_n - x| \leq 2^{-n+1} \) for every \( n \in \mathbb{N} \), i.e.

\[
A \subseteq B_{2^{-n+1}} (A_n) \text{ for every } n \in \mathbb{N},
\]

i.e. \( A \) is bounded, and so \( \overline{A} \in A \). Let \( \varepsilon > 0 \) be arbitrary, then we find a natural number \( N \in \mathbb{N} \) such that

\[
\varepsilon > 2^{-N}.
\]

Let \( n \in \mathbb{N} \) be with \( n > N + 1 \). For every \( x_n \in A_n \) we find an element \( x \in \overline{A} \) with

\[
|x_n - x| \leq 2^{-n+1} \leq 2^{-(N+1)+1} = 2^{-N} < \varepsilon,
\]

i.e.

\[
A_n \subseteq B_{2^{-n+1}} (\overline{A}).
\]

This implies

\[
d_H (A_n, \overline{A}) < 2^{-n+1} < \varepsilon \text{ for all } n \in \mathbb{N} \text{ with } n \geq N + 1,
\]

i.e. \( (A_n)_{n \in \mathbb{N}} \) converge to \( \overline{A} \).

Now it’s left to show that the set \( A \) is closed under above construction. Let \( a \in \overline{A} \) be an arbitrary limit point and choose \( (a_k)_{k \in \mathbb{N}} \subseteq A \) such that \( \lim_{k \to \infty} a_k = a \) in \( M \). Then by construction we find for every \( k \in \mathbb{N} \) a sequence \( (y_{nk}^{(k)})_{n \in \mathbb{N}} \) with \( y_{nk}^{(k)} \in A_n \) and \( \lim_{n \to \infty} y_{nk}^{(k)} = a_k \). Now choose \( n_1 \in \mathbb{N} \) such that \( |y_{n_1}^{(1)} - a_1| < 1 \). Do it again choose \( n_2 > n_1 \) with \( |y_{n_2}^{(1)} - a_2| < \frac{1}{2} \). Repeat this prozess, then we get a strictly monotone increasing sequence \( (n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N} \) with \( y_{nk}^{(k)} \in A_{n_k} \) and \( |y_{n_k}^{(k)} - a_k| < \frac{1}{k} \) for all \( k \in \mathbb{N} \). By triangle-inequality it follows:

\[
|y_{nk}^{(k)} - a| \leq |y_{nk}^{(k)} - a_k| + |a_k - a| < \frac{1}{k} + \varepsilon
\]

for \( k \in \mathbb{N} \) large enough, i.e.

\[
\lim_{k \to \infty} y_{nk}^{(k)} = a.
\]

Since every convergent sequence is a Cauchy-sequence, the Extension Lemma gives us a Cauchy-sequence \( (x_n)_{n \in \mathbb{N}} \subseteq M \) with \( x_n \in A_n \) and \( x_{nk} = y_{nk}^{(k)} \) for every \( n, k \in \mathbb{N} \). This conludes \( \lim_{n \to \infty} x_n = a \) and that means \( a \in A \), i.e. \( A \) is closed, since \( a \) was arbitrary. Everything together we see that \( K (M) \) is complete.

(3) Let \( M \) be compact, then we know by definition that \( M \) is totally bounded and complete. This implies with (1) and (2) that \( K (M) \) is also totally bounded and complete. By definition of compactness again we know that \( K (M) \) is compact.

Exercise 2 (The Ehrling Lemma)

Let \( X \) be a normed space with three norms \( \| \cdot \|_a, \| \cdot \|_b \) and \( \| \cdot \|_c \). These norms have the following two properties:

1. For every sequence in \( X \) which is bounded in the \( \| \cdot \|_a \)-norm there is a subsequence which converges with respect to \( \| \cdot \|_b \)-norm.
2. There is a constant \( \Lambda > 0 \) such that

\[
\|x\|_c \leq \Lambda \|x\|_b \text{ for all } x \in X.
\]
Show that for every $\epsilon > 0$ there is a constant $C_{\epsilon} > 0$ such that
\[ \|x\|_b \leq \epsilon \|x\|_a + C_{\epsilon} \|x\|_c \] for all $x \in X$.

**Exercise 3** (Precompactness in $C^0(I, K)$)

Let $I := [0, 1]$ be the unit interval in $\mathbb{R}$. Which of the following families is precompact in $C^0(I, K)$ with $K \in \{\mathbb{R}, \mathbb{C}\}$ with respect to the sup-norm?

1. $A := \{f_n : n \in \mathbb{N}\}$ with $f_n(x) = \sin(x + n)$ for $x \in I$ and $n \in \mathbb{N}$.
2. $B := \{f_n : n \in \mathbb{N}\}$ with $f_n(x) = \sin(nx)$ for $x \in I$ and $n \in \mathbb{N}$.

**Exercise 4** (Relativ compactness)

Set $I := [0, 1] \subseteq \mathbb{R}$ and let $k \in C^0(I \times I, \mathbb{K})$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be a continuous function on $I \times I$. We define the integral operator $T : (C^0(I, \mathbb{K}), \|\cdot\|_{C^0(I, \mathbb{K})}) \to (C^0(I, \mathbb{K}), \|\cdot\|_{C^0(I, \mathbb{K})})$ with kernel $k$ by
\[ Tf(x) = \int_0^1 k(x, y)f(y)dy, \quad x \in I. \]

On Exercise sheet 2, Exercise 3 you saw that $T$ is a well-defined linear and continuous/bounded operator on $X := C^0(I, \mathbb{K})$. Show that the set
\[ K := \{Tf : f \in B^\mathbb{X}_r(0)\} \]
is relativ compact in $X$, i.e. the closure of $K$ is compact in $X$, where $B^\mathbb{X}_r(g)$ is the ball in $X$ with radius $r > 0$ and center $g \in X$.

**Solution of Exercise 4**

We want to use the theorem of Arzela-Ascoli. The interval $I$ is obviously compact in $\mathbb{R}$, since it’s closed and bounded (Heine-Borel). The family $K$ is pointwise bounded, since for any $x \in I$, $g \in K$ we find a function $f \in B^\mathbb{X}_1(0)$ with $Tf = g$ and then we get
\[ |Tf(x)| = \left| \int_0^1 k(x, y)f(y)dy \right| \leq \int_0^1 |k(x, y)| \cdot |f(y)|dy \]
\[ \leq \int_0^1 \|k\|_{C^0(I \times I, \mathbb{K})} \cdot \|f\|_{C^0(I, \mathbb{K})} dy = \|k\|_{C^0(I \times I, \mathbb{K})} \cdot \|f\|_{C^0(I, \mathbb{K})} \leq \|k\|_{C^0(I \times I, \mathbb{K})} \cdot \|f\|_{C^0(I, \mathbb{K})} \]
by triangle-inequality. The family $K$ is uniformly equicontinuous, since for every $\epsilon > 0$ we find with the uniform continuity of $k$ on $I \times I$ a $\delta(\epsilon) :=: \delta > 0$ such that for pairs $(x, y), (x', y') \in I \times I$ with $|(x, y) - (x', y')| < \delta$ we have
\[ |k(x, y) - k(x', y')| < \epsilon. \]
Then we get for $g \in K$ and $f \in B^\mathbb{X}_1(0)$ with $Tf = g$:
\[ |Tf(x) - Tf(x')| = \left| \int_0^1 (k(x, y) - k(x', y))f(y)dy \right| \leq \int_0^1 |k(x, y) - k(x', y)| \cdot |f(y)|dy \]
\[ \leq \int_0^1 |k(x, y) - k(x', y)| \cdot |f(y)|dy \leq \int_0^1 |k(x, y) - k(x', y)| dy \|f\|_{C^0(I, \mathbb{K})} \]
\[ \leq \int_0^1 \epsilon dy \|f\|_{C^0(I, \mathbb{K})} \leq \epsilon \]
for all $x, x' \in I$ with $|x - x'| < \delta$, i.e. the family $K$ is uniformly equicontinuous. The theorem of Arzela-Ascoli implies now, that the set $K$ is precompact in $C^0(I, \mathbb{K})$ with respect to the sup-norm.