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Functional analysis

Solutions to 4. Exercise Sheet

Exercise 1 (Application of the Hahn-Banach Theorem)

Find a functional $\Phi \in l^\infty(\mathbb{N}, \mathbb{R})'$ with $\|\Phi\| = 1$ and the following three properties:

(1) $\Phi(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i$ for $x \in l^\infty(\mathbb{N}, \mathbb{R})$, if this limit exists.

(2) $\liminf_{i\to\infty} x_i \le \Phi(x) \le \limsup_{i\to\infty} x_i$ for $x \in l^\infty(\mathbb{N}, \mathbb{R})$.

(3) $\Phi((x_1, x_2, x_3, \ldots)) = \Phi((x_2, x_3, \ldots))$ for $x = (x_1, x_2, x_3, \ldots) \in l^\infty(\mathbb{N}, \mathbb{R})$.

Exercise 2 ((C) Projection to finite-dimensional subspaces)

Let $(X, \|\cdot\|_X)$ be a normed space and $U \subseteq X$ be a finite-dimensional subspace of X. We call a linear map $P\colon X \to X$ a Projection if and only if $P^2 = P$ on X. Construct a continuous Projection P with $\operatorname{Im}(P) = U$ and conclude that the equality

$$X = \ker(P) \oplus U$$

holds.

Exercise 3 ((C) Separability)

Show that every subset A of a separable metric space (X, d_X) is also separable (with the induced metric). Conclude that the space $(l^\infty(\mathbb{N}, \mathbb{R}), \|\cdot\|_{l^\infty})$ is not separable.

Exercise 4 (Product of Hölder-continuous functions)

Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with some chord-arc condition and $k \in \mathbb{N}_0, \alpha \in (0,1]$ be arbitrary. Show that if $u, v \in C^{k,\alpha}\left(\overline{\Omega}\right)$ then the product $u \cdot v$ is also in $C^{k,\alpha}\left(\overline{\Omega}\right)$ with norm estimate

$$\|u \cdot v\|_{C^{k,\alpha}(\Omega)} \le C(k, \Omega) \|u\|_{C^{k,\alpha}(\Omega)} \|v\|_{C^{k,\alpha}(\Omega)} \text{ for some constant } C(k,\Omega) > 0.$$

Solution of Exercise 4

We will do an induction over the index k.
I.B. $k = 0$: We get for $x, y \in \Omega$, $x \neq y$, with triangle-inequality

$$\begin{aligned}
\frac{|(uv)(x) - (uv)(y)|}{|x-y|^\alpha} &= \frac{|(u(x) - u(y))\, v(x) + u(y)\,(v(x) - v(y))|}{|x-y|^\alpha} \\
&\le \frac{|(u(x) - u(y))\, v(x)|}{|x-y|^\alpha} + \frac{|u(y)\,(v(x) - v(y))|}{|x-y|^\alpha} \\
&\le \frac{|u(x) - u(y)|}{|x-y|^\alpha} \|v\|_{C^0(\Omega)} + \frac{|v(x) - v(y)|}{|x-y|^\alpha} \|u\|_{C^0(\Omega)} \\
&\le [u]_{\alpha,\Omega} \|v\|_{C^0(\Omega)} + [v]_{\alpha,\Omega} \|u\|_{C^0(\Omega)} \\
&\le \|u\|_{C^{0,\alpha}(\Omega)} \|v\|_{C^{0,\alpha}(\Omega)} + \|v\|_{C^{0,\alpha}(\Omega)} \|u\|_{C^{0,\alpha}(\Omega)} = 2 \|u\|_{C^{0,\alpha}(\Omega)} \|v\|_{C^{0,\alpha}(\Omega)}.
\end{aligned}$$

This implies now

$$\|uv\|_{C^{k,\alpha}(\Omega)} = \|uv\|_{C^0(\Omega)} + [uv]_{\alpha,\Omega} \le \|u\|_{C^0(\Omega)} \|v\|_{C^0(\Omega)} + [uv]_{\alpha,\Omega}$$

$$\leq \|u\|_{C^{0,\alpha}(\Omega)} \|v\|_{C^{0,\alpha}(\Omega)} + 2 \|u\|_{C^{0,\alpha}(\Omega)} \|v\|_{C^{0,\alpha}(\Omega)} = 3 \|u\|_{C^{0,\alpha}(\Omega)} \|v\|_{C^{0,\alpha}(\Omega)} .$$

I.H.: For $k \in \mathbb{N}_0$ fixed, but arbitrary we have the estimate for all $l \in \mathbb{N}_0$ with $l \leq k$:

$$\|u \cdot v\|_{C^{l,\alpha}(\Omega)} \leq C(l, \Omega) \|u\|_{C^{l,\alpha}(\Omega)} \|v\|_{C^{l,\alpha}(\Omega)}$$

for a constant $C(l, \Omega) > 0$ and for all $u, v \in C^{l,\alpha}(\Omega)$.
I.S.: Let $\gamma \in \mathbb{N}_0^n$ be a multiindex with $|\gamma| = k + 1$. Then we find some index $i \in \{1, \ldots, n\}$ and some multiindex $\gamma' \in \mathbb{N}_0^n$ with $|\gamma'| = k$ and $\gamma = \gamma' + e_i$. So we have

$$\begin{aligned}
\|D^\gamma(uv)\|_{C^{0,\alpha}(\Omega)} &= \left\|\partial_i D^{\gamma'}(uv)\right\|_{C^{0,\alpha}(\Omega)} = \left\|D^{\gamma'}\partial_i(uv)\right\|_{C^{0,\alpha}(\Omega)} \\
&= \left\|D^{\gamma'}\left((\partial_i u)\, v + u\,(\partial_i v)\right)\right\|_{C^{0,\alpha}(\Omega)} \\
&\leq \left\|D^{\gamma'}(\partial_i u)\, v\right\|_{C^{0,\alpha}(\Omega)} + \left\|D^{\gamma'} u\,(\partial_i v)\right\|_{C^{0,\alpha}(\Omega)} \\
&\leq C(k, \Omega) \left(\|(\partial_i u)\, v\|_{C^{k,\alpha}(\Omega)} + \|u\,(\partial_i v)\|_{C^{k,\alpha}(\Omega)}\right) \\
&\leq C(k, \Omega) \left(\|\partial_i u\|_{C^{k,\alpha}(\Omega)} \|v\|_{C^{k,\alpha}(\Omega)} + \|u\|_{C^{k,\alpha}(\Omega)} \|\partial_i v\|_{C^{k,\alpha}(\Omega)}\right) \\
&\leq C(\kappa, \operatorname{diam}(\Omega))\, C(k, \Omega) \left(\|u\|_{C^{k+1,\alpha}(\Omega)} \|v\|_{C^{k+1,\alpha}(\Omega)} + \|u\|_{C^{k+1,\alpha}(\Omega)} \|v\|_{C^{k+1,\alpha}(\Omega)}\right) \\
&= 2C(\kappa, \operatorname{diam}(\Omega))\, C(k, \Omega) \|u\|_{C^{k+1,\alpha}(\Omega)} \|v\|_{C^{k+1,\alpha}(\Omega)} ,
\end{aligned}$$

since the space $C^{k+1,\alpha}\left(\overline{\Omega}\right)$ is compact embedded in the space $C^{k,\alpha}\left(\overline{\Omega}\right)$. This implies now

$$\|uv\|_{C^{k+1,\alpha}(\Omega)} \leq C(k+1, \Omega) \|u\|_{C^{k+1,\alpha}(\Omega)} \|v\|_{C^{k+1,\alpha}(\Omega)} ,$$

where $C(k + 1, \Omega) > 0$ depends only on k, the diameter of Ω ($\operatorname{diam}(\Omega)$) and the chord-arc-constant κ of Ω. $\square$