Functional analysis

5. Exercise Sheet

Exercise 1  ((C) About Bounded Inverse)
Find an example for a continuous operator $T \in L(X, X)$ on a normed space $(X, \|\cdot\|_X)$ such that the inverse $T^{-1}$ is not continuous.

Exercise 2  (Continuity of bilinear forms)
Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be two Banach spaces and $B : X \times Y \to \mathbb{R}$ be a bilinear form such that for every $x \in X$ resp. $y \in Y$ the map $B(x, \cdot)$ resp. $B(\cdot, y)$ is continuous. Show that $B$ is continuous on $(X \times Y, \|\cdot\|_{X \times Y})$ with $\|(x, y)\|_{X \times Y} := \|x\|_X + \|y\|_Y$ for $(x, y) \in X \times Y$.

Exercise 3  ((C) Pointwise convergent operators)
Let $(X, \|\cdot\|_X)$ be a Banach space and $(Y, \|\cdot\|_Y)$ a normed space and $(T_n)_{n \in \mathbb{N}} \subseteq L(X, Y)$ be a pointwise convergent operator sequence, i.e. for every $x \in X$ there is an element $Tx \in Y$ with $Tx = \lim_{n \to \infty} T_n x$ in $Y$.

(1) Show that the operator $T \in L(X, Y)$ and $\|T\| \leq \liminf_{n \to \infty} \|T_n\|$.

(2) Find an example for “<” in the inequality in (1).

Exercise 4  (Commutators)
Let $(X, \|\cdot\|_X)$ be a normed space and let $P, Q : X \to X$ be linear with $[P, Q] := PQ - QP = \text{Id}_X$. Show that either $P$ or $Q$ is not continuous on $X$.
(Hint: Show first that $PQ^{n+1} - Q^{n+1}P = (n + 1)Q^n$ holds for all $n \in \mathbb{N}$.)