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Functional analysis

6. Exercise Sheet

Exercise 1 ((C) Canonical Embedding of L^1 in $(L^\infty)'$)

Show that the map

$$J\colon L^1(\mathbb{R}) \to L^\infty(\mathbb{R})',\ J(f)(g) = \int_{\mathbb{R}} fg$$

is not surjective. (Hint: Construct a functional $\varphi \in L^\infty(\mathbb{R})'$ with $\varphi(g) = g(0)$ for every $g \in L^\infty(\mathbb{R}) \cap C^0(\mathbb{R})$.)

Exercise 2 (Convergence of L^p-Norm)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, i.e. $\mathbb{P}(\Omega) = 1$, and $u\colon \Omega \to \mathbb{R}$ be $\mathbb{P}$-measurable. What are the limits, if they exist, of the function

$$\Phi\colon (-\infty, \infty)\backslash\{0\} \to \mathbb{R},\ \Phi(p) = \left(\int_\Omega |u|^p \,\mathrm{d}\mathbb{P}\right)^{\frac{1}{p}}$$

for $p \to \infty$ and for $p \to -\infty$.

Exercise 3 ((C) Closable operators)

Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be two Banach spaces and $T\colon D(T) \to Y$ be a linear map with domain $D(T) \subseteq X$, $D(T)$ is a linear subspace of X. The operator T is called closable if and only if $\overline{\operatorname{graph}(T)}$ is again the graph of a linear operator $\overline{T}$. We call T closed if and only if T is closeable with $\overline{T} = T$.

(1) Show: T is closable if and only if for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq D(T)$ with $x_n \to 0$ in X and $Tx_n \to y \in Y$ in Y for $n \to \infty$ we have $y = 0$.

(2) Now we have $X = Y = L^2(\Omega)$ with some open set $\Omega \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, and the differential operator of order $k \in \mathbb{N}_0$

$$T = \sum_{\alpha\in\mathbb{N}_0^d\colon |\alpha|\leq k} a_\alpha D^\alpha \text{ with coefficients } a_\alpha \in C^\infty(\Omega),$$

where the domain of T is

$$D(T) := \left\{u \in C^\infty\colon u, Tu \in L^2(\Omega)\right\}.$$

Show that the operator T is closable.

Exercise 4 (L^∞ is not separable)

Let $(X, \|\cdot\|_X)$ be a Banach space and we assume that there is a family $(O_j)_{j\in I} \subseteq X$ such that

(i) For each $i \in I$ the set O_i is a nonempty open subset of X

(ii) Disjointness: $O_i \cap O_j = \emptyset$ for all $i, j \in I$ with $i \neq j$

(iii) The set I is uncountable.

Prove that the space X is not separable and conclude afterwards that $L^\infty(\mathbb{R}^d)$, $d \in \mathbb{N}$, is not separable.