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Prof. Dr. Tobias Lamm
M.Sc. Michael Ullmann

Karlsruhe Institute of Technology

Functional analysis

7. Exercise Sheet

Exercise 1 ((C) Dual spaces of c_0 and c)

What are the dual spaces of

$$c_0 := \left\{ x = (x_1, x_2, x_3, \dots) \in l^\infty(\mathbb{N}, \mathbb{R}) : \lim_{n\to\infty} x_n = 0 \right\},$$
$$c := \left\{ x = (x_1, x_2, x_3, \dots) \in l^\infty(\mathbb{N}, \mathbb{R}) : \lim_{n\to\infty} x_n \text{ exists in } \mathbb{R} \right\}$$

endowed with the sup-norm $\|\cdot\|_{l^\infty(\mathbb{N})}$?

Exercise 2 ((C) Characterisation of strictly normed $\mathbb{R}$-spaces)

Let $(X, \|\cdot\|_X)$ be a normed $\mathbb{R}$-vectorspace. Show that the following are equivalent:

(1) X is strictly normed, i.e. for all $x, y \in X$ we have

$$\|x+y\|_X = \|x\|_X + \|y\|_X \;\Rightarrow\; x, y \text{ are linearly dependent.}$$

(2) The unit ball $B_1(0) \subseteq X$ is strictly convex, i.e. for all $x, y \in X$ we have

$$\|x\|_X = \|y\|_X = 1,\; x \neq y \;\Rightarrow\; \left\| \frac{x+y}{2} \right\|_X < 1.$$

(3) Every closed convex subset $K \subseteq X$ has at most one element with minimal norm.

Exercise 3 (Hausdorff measures are Borel-regular)

Show that the Hausdorff measure $\mathcal{H}^s$, $s \geq 0$, on a metric space (X, d_X) is a Borel-regular outer measure where we define for all subsets $A \subseteq X$ and $\delta > 0$ the outer measures

$$\mathcal{H}^s_\delta(A) := \inf \left\{ \sum_{n=1}^\infty \operatorname{diam}(A_n)^s : A \subseteq \bigcup_{n=1}^\infty A_n \text{ and } \operatorname{diam}(A_n) \leq \delta \text{ for all } n \in \mathbb{N} \right\},\; \inf \emptyset := \infty,$$
$$\mathcal{H}^s(A) := \sup_{\delta > 0} \mathcal{H}^s_\delta(A).$$

Exercise 4 ($C^0_c(X, \mathbb{R})$ is dense in $L^1(X, \mu)$)

Let (X, d_X) be a σ-compact metric space. Show that the space $C^0_c(X, \mathbb{R})$ is dense in $L^1(X, \mu)$ for all Radon measures μ on X.
(Hint: Use the fact that step-functions are dense in $L^1(X, \mu)$.)