Functional analysis

Solutions to 7. Exercise Sheet

Exercise 1 ((C) Dual spaces of \(c_0\) and \(c\))

What are the dual spaces of \(c_0 := \{ x = (x_1, x_2, x_3, \ldots) \in l^\infty (\mathbb{N}, \mathbb{R}) : \lim_{n \to \infty} x_n = 0 \} \),
\[ \text{c} := \{ x = (x_1, x_2, x_3, \ldots) \in l^\infty (\mathbb{N}, \mathbb{R}) : \lim_{n \to \infty} x \text{ exists in} \mathbb{R} \} \]
endowed with the sup-norm \( \| \cdot \|_{l^\infty(\mathbb{N})} \)?

Exercise 2 ((C) Characterisation of strictly normed \(\mathbb{R}\)-spaces)

Let \((X, \| \cdot \|_X)\) be a normed \(\mathbb{R}\)-vectorspace. Show that the following are equivalent:

1) \(X\) is strictly normed, i.e. for all \(x, y \in X\) we have
\[ \| x + y \|_X = \| x \|_X + \| y \|_X \Rightarrow x, y \text{ are linearly dependent.} \]

2) The unit ball \(B_1(0) \subseteq X\) is strictly convex, i.e. for all \(x, y \in X\) we have
\[ \| x \|_X = \| y \|_X = 1, x \neq y \Rightarrow \left\| \frac{x + y}{2} \right\|_X < 1. \]

3) Every closed convex subset \(K \subseteq X\) has at most one element with minimal norm.

Solution of Exercise 2

(1)⇒(2): Let \(x, y \in X\) with \(\| x \|_X = 1 = \| y \|_X\) and \(x \neq y\) in \(X\). If we assume that \(\left\| \frac{x + y}{2} \right\|_X \geq 1\) we get
\[ 2 \leq \| x + y \|_X \leq \| x \|_X + \| y \|_X = 1 + 1 = 2, \text{ i.e. } \| x + y \|_X = 2 = \| x \|_X + \| y \|_X. \]
The property (1) implies now that \(x\) and \(y\) are linear dependent, so we find some number \(\lambda \in \mathbb{K}\) such that \(x = \lambda y\). But then we have that
\[ 2 = \| x + y \|_X = \| x + \lambda x \|_X = |1 + \lambda| \| x \|_X, \text{ and } 1 = \| y \|_X = \| \lambda x \|_X = |\lambda| \| x \|_X = |\lambda|, \]
i.e. \(\lambda = 1\). Then we get that \(x = y\), which is a contradiction to \(x \neq y\), i.e. our assumption was wrong and (2) holds.

(2)⇒(3): Let \(K \subseteq X\) be a closed and convex subset of \(X\) and we assume w.l.o.g. that there is some element in \(K\) with minimal norm. Let \(y, z \in K\) be two elements with minimal norm in \(K\) and set
\[ r := \inf_{x \in K} \| x \|_X, \text{ i.e. } \| y \|_X = \| z \|_X = r. \]
If \(r = 0\), then \(\| y \|_X = \| z \|_X = 0\) and that means \(y = z = 0\) in \(X\). Hence let \(r > 0\). Since \(K\) is convex we know that \(\frac{y + z}{2} \in K\), and we have for the norm:
\[ \left\| \frac{y + z}{2} \right\|_X \geq r. \]
If \(y \neq z\) in \(X\), we can apply property (2) for \(\frac{y}{r}\) and for \(\frac{z}{r}\) and we get
\[ 1 > \left\| \frac{y}{r} + \frac{z}{r} \right\|_X = \left\| \frac{y + z}{2r} \right\|_X = \frac{1}{r} \left\| \frac{y + z}{2} \right\|_X \Leftrightarrow r > \left\| \frac{y + z}{2} \right\|_X, \]

which is a contradiction to
\[ \left\| \frac{y + z}{2} \right\|_X \geq r, \]
i.e. assumption was wrong and \( y = z \) in \( X \), i.e. (3) holds.

(3)⇒(1): Let \( x, y \in X \) be linearly independent, especially \( x \neq 0 \neq y \). We take the normed elements:
\[ \tilde{x} := \frac{x}{\|x\|_X} \quad \text{and} \quad \tilde{y} := \frac{y}{\|y\|_X}. \]

And we define the subset
\[ K := \{(1 - s)\tilde{x} + s\tilde{y} : s \in [0, 1]\} \subseteq X. \]
As a set of all convex-combinations of \( \tilde{x}, \tilde{y} \) the subset \( K \) is convex. Let \((z_n)_{n \in \mathbb{N}} \subseteq K\) be a converging sequence to an element \( z \in X \) in \( X \). Then we find for every \( n \in \mathbb{N} \) some \( s_n \in [0, 1] \) such that
\[ z_n = (1 - s_n)\tilde{x} + s_n\tilde{y}. \]
Since the sequence \((s_n)_{n \in \mathbb{N}}\) is bounded we get with the theorem of Bolzano-Weierstraß a converging subsequence \((s_{n_k})_{k \in \mathbb{N}}\) to some \( s \in [0, 1] \), and that implies that
\[ z = \lim_{n \to \infty} z_n = \lim_{k \to \infty} [(1 - s_{n_k})\tilde{x} + s_{n_k}\tilde{y}] = (1 - s)\tilde{x} + s\tilde{y} \in K, \]
i.e. \( K \) is closed. Now we take the functional \( \varphi : [0, 1] \to [0, \infty) \), \( s \mapsto \| (1 - s)\tilde{x} + s\tilde{y} \|_X \). This functional is continuous as a composition of continuous function and \( \varphi \) is convex, since for some \( a, b, t \in [0, 1] \) we have that
\[ \varphi(ta + (1 - t)b) = \| (1 - ta - (1 - t)b)\tilde{x} + (ta + (1 - t)b)\tilde{y} \|_X \]
\[ = \| t(1 - a)\tilde{x} + ta \tilde{y} + (1 - t)(1 - b)\tilde{x} + (1 - t)b\tilde{y} \|_X \]
\[ \leq t\| (1 - a)\tilde{x} + a\tilde{y} \|_X + (1 - t)\| (1 - b)\tilde{x} + b\tilde{y} \|_X \]
\[ = t\varphi(a) + (1 - t)\varphi(b). \]

Since \( \varphi \) is continuous on \([0, 1]\) and \([0, 1]\) is compact in \( \mathbb{R} \) there exists a point \( s_0 \in [0, 1] \) with \( \varphi(s_0) = \min_{s \in [0, 1]} \varphi(s) \geq 0 \), i.e. in \( K \) there is some element \((1 - s_0)\tilde{x} + s_0\tilde{y} \in K\) with minimal norm. Property (3) gives us that this element is unique, and since
\[ \varphi(0) = \| (1 - 1)\tilde{x} + 1\tilde{y} \|_X = \| \tilde{y} \|_X = 1 = \| \tilde{x} \|_X = \| (1 - 0)\tilde{x} + 0\tilde{y} \|_X = \varphi(0), \]
we know that \( s_0 \in (0, 1) \) and \( \varphi(s_0) < 0 \). In general we have by triangle inequality:
\[ 0 \leq \varphi(s) = \| (1 - s)\tilde{x} + s\tilde{y} \|_X \leq (1 - s) \| \tilde{x} \|_X + s \| \tilde{y} \|_X = (1 - s) + s = 1 \text{ for all } s \in [0, 1]. \]
And for some \( s' \in (0, 1) \) there are \( t, s^* \in [0, 1] \) such that \( s = (1 - t)s_0 + ts^* \) then we get since \( \varphi \) is convex
\[ \varphi(s) = \varphi((1 - t)s_0 + ts^*) \leq (1 - t)\varphi(s_0) + t\varphi(s^*) \leq (1 - t)\varphi(s_0) + t \cdot 1 < (1 - t) \cdot 1 + 1 = 1, \]
i.e. \( \varphi < 1 \) on \((0, 1)\). We set \( r := \frac{\|x\|_X + \|y\|_X}{2} > 0 \) and
\[ 0 < s := \frac{\|y\|_X}{2r} = \frac{\|y\|_X}{\|x\|_X + \|y\|_X} < 1. \]

Then we have that
\[ r \left( (1 - s)\tilde{x} + s\tilde{y} \right) = r \left( 1 - \frac{\|y\|_X}{2r} \right) \left( \frac{x}{\|x\|_X} \right) + \frac{\|y\|_X}{2r} \left( \frac{y}{\|y\|_X} \right) \]
\[ = r \left( \frac{2r - \|y\|_X}{2r} \right) \frac{x}{\|x\|_X} + \frac{\|y\|_X}{2r} \frac{y}{\|y\|_X} \]
\[ = \frac{\|x\|_X + \|y\|_X}{2} \left( \frac{x}{\|x\|_X} + \frac{y}{\|y\|_X} \right) = \frac{x}{2} + \frac{y}{2} = \frac{x + y}{2}. \]

This implies that
\[ 1 > \varphi(s) = \frac{x + y}{2r} \quad \iff \quad \|x + y\|_X < 2r = \|x\|_X + \|y\|_X, \]
i.e. (1) holds.
This proves the equivalence. \(\square\)
Exercise 3  (Hausdorff measures are Borel-regular)

Show that the Hausdorff measure $\mathcal{H}^s$, $s \geq 0$, on a metric space $(X, d_X)$ is a Borel-regular outer measure where we define for all subsets $A \subseteq X$ and $\delta > 0$ the outer measures

$$
\mathcal{H}^\delta_A(A) := \inf \left\{ \sum_{n=1}^{\infty} \text{diam} (A_n)^s : A \subseteq \bigcup_{n=1}^{\infty} A_n \text{ and diam} (A_n) \leq \delta \text{ for all } n \in \mathbb{N} \right\}, \quad \inf \emptyset := \infty,
$$

$$
\mathcal{H}^s(A) := \sup_{\delta > 0} \mathcal{H}^\delta_A(A).
$$

Solution of Exercise 3

(4) $\mathcal{H}^s$ is Borel regular.

For any $C \subseteq X$ we have that $\text{diam}(C) = \text{diam}(\overline{C})$, since for $x, y \in \overline{C}$ we can choose sequences $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subseteq C$ with $x_n \to x$ and $y_n \to y$ in $X$, then we have

$$
d_X(x, y) = \lim_{n \to \infty} d_X(x_n, y_n) \leq \lim_{n \to \infty} \text{diam}(C) = \text{diam}(C),
$$

i.e.

$$
\text{diam}(C) \leq \text{diam}(\overline{C}) \leq \text{diam}(C).
$$

Therefore we can also write

$$
\mathcal{H}^\delta_A(A) = \inf \left\{ \sum_{n=1}^{\infty} \text{diam} (A_n)^s : A \subseteq \bigcup_{n=1}^{\infty} A_n, \text{ diam} (A_n) \leq \delta, \ A_n \subseteq X \text{ is closed} \right\}
$$

for all $A \subseteq X$. Let $A \subseteq X$ with $\mathcal{H}^\delta(A) < \infty$, otherwise it is trivial, since $X$ is a Borel set with $\mathcal{H}^s(X) = \infty$. We know that $\mathcal{H}^\delta(A) < \infty$ for all $\delta > 0$. For every $k \in \mathbb{N}$ choose a sequence $(C_n^{(k)})_{n \in \mathbb{N}} \subseteq X$ such that $\text{diam} \left( C_n^{(k)} \right) \leq \frac{1}{k}$ with $A \subseteq \bigcup_{n=1}^{\infty} C_n^{(k)}$ and with

$$
\sum_{n=1}^{\infty} \text{diam} \left( C_n^{(k)} \right) \leq \mathcal{H}^\delta_A(A) + \frac{1}{k}.
$$

We set for $k \in \mathbb{N}$

$$
A_k := \bigcup_{n=1}^{\infty} C_n^{(k)}, \quad B := \bigcap_{k=1}^{\infty} A_k.
$$

Then the subset $B$ is Borel (by definition) and $A \subseteq A_k$ for all $k \in \mathbb{N}$, i.e. $A \subseteq B$. We get that

$$
\mathcal{H}^s(B) \leq \sum_{n=1}^{\infty} \text{diam} \left( C_n^{(k)} \right) \leq \mathcal{H}^\delta_A(A) + \frac{1}{k} \text{ for all } k \in \mathbb{N}.
$$

For $k \to \infty$ we get now

$$
\mathcal{H}^s(B) \leq \mathcal{H}^s(A) \leq \mathcal{H}^s(B), \text{ since } A \subseteq B,
$$

i.e.

$$
\mathcal{H}^s(A) = \mathcal{H}^s(B).
$$

This means that the Borel outer measure $\mathcal{H}^s$ is Borel regular for all $s \geq 0$.  

Exercise 4  ($C_0^0(X, \mathbb{R})$ is dense in $L^1(X, \mu)$)

Let $(X, d_X)$ be a $\sigma$-compact metric space. Show that the space $C_0^0(X, \mathbb{R})$ is dense in $L^1(X, \mu)$ for all Radon measures $\mu$ on $X$.

(Hint: Use the fact that step-functions are dense in $L^1(X, \mu)$.)

Solution of Exercise 4

Step 01.: Nonnegative functions $f \in L^1(X, \mu)$ can be approximated by stepfunctions.

Let $f \in L^1(X, \mu)$ be a nonnegative function, i.e. $f \geq 0$ on $X$. We define for all $k, n \in \mathbb{N}_0$ the set

$$
A_{k,n} := \left\{ x \in X : 2^{-n}k \leq f(x) < 2^{-n}(k+1) \right\},
$$
and the stepfunctions

\[ s_n := \sum_{k=0}^{2^n} 2^{-n}k\chi_{A_{k,n}} \geq 0 \text{ on } X, \]

where the characteristic function \( \chi_B : X \to \{0,1\} \) for some subset \( B \subseteq X \) is defined by

\[ \chi_B(x) := \begin{cases} 1, & x \in B \\ 0, & x \notin B \end{cases} \text{ for } x \in X. \]

It is for \( n \in \mathbb{N}_0 \)

\[ B_n := \bigcup_{k=0}^{2^n} A_{k,n} = \{ x \in X : 0 \leq f(x) < 2^n + 2^{-n} \}, \quad X = \bigcup_{n=1}^{\infty} B_n. \]

If we take some point \( x \in B_n, n \in \mathbb{N}_0 \), then we find some unique \( k_0 \in \mathbb{N}_0 \) such that \( x \in A_{k_0,n} \), it follows:

\[ |f(x) - s_n(x)| = \left| f(x) - \sum_{k=0}^{2^n} 2^{-n}k\chi_{A_{k,n}}(x) \right| = \left| f(x) - 2^{-n}k_0\chi_{A_{k_0,n}}(x) \right| = |f(x) - 2^{-n}k_0| = f(x) - 2^{-n}k_0 < 2^{-n} (k_0 + 1) - 2^{-n}k_0 = 2^{-n} \to 0 \text{ for } n \to \infty, \]

i.e. \( (s_n)_{n\in\mathbb{N}} \) converge pointwise to \( f \) on \( X \). Since the function \( f \in L^1(X,\mu) \), we know that \( s_n \in L^1(X,\mu) \) for \( n \in \mathbb{N} \), because

\[ |s_n(x)| = \sum_{k=0}^{2^n} 2^{-n}k\chi_{A_{k,n}} < \sum_{k=0}^{2^n} f(x)\chi_{A_{k,n}} \]

\[ = f(x)\chi_{\bigcup_{k=0}^{2^n} A_{k,n}}(x) = f(x)\chi_{B_n}(x) \]

\[ \|s_n\|_{L^1(X,\mu)} = \int_X |s_n(x)|d\mu(x) \leq \int_X f(x)\chi_{B_n}(x)d\mu(x) = \int_{B_n} f(x)d\mu(x) \]

\[ \leq \int_X |f(x)|d\mu(x) = \|f\|_{L^1(X,\mu)} < \infty. \]

The Theorem of Lebesgue gives us that the sequence \( (s_n)_{n\in\mathbb{N}} \) converge to the function \( f \) in \( L^1(X,\mu) \), i.e.

\[ \lim_{n \to \infty} \|f - s_n\|_{L^1(X,\mu)} = \lim_{n \to \infty} \int_X |f(x) - s_n(x)|d\mu(x) = \lim_{n \to \infty} \int_X (f(x) - s_n(x))d\mu(x) = 0. \]

Step 02.: Functions \( f \in L^1(X,\mu) \) can be approximated by stepfunctions.

We decompose \( f \) in positive and negative part, i.e. \( f = f^+ - f^- \) with

\[ f^+(x) := \sup \{ 0, f(x) \}, \quad f^-(x) := \sup \{ 0, -f(x) \} \text{ for } x \in X. \]

Then we have that \( f^+, f^- \geq 0 \) on \( X \) and \( f^+, f^- \in L^1(X,\mu) \), since we have

\[ \|f^\pm\|_{L^1(X,\mu)} = \int_X |f^\pm(x)|d\mu(x) \leq \int_X |f(x)|d\mu(x) = \|f\|_{L^1(X,\mu)} < \infty. \]

By step 01. there are two sequences of stepfunctions \( (s_{n}^{(1)})_{n\in\mathbb{N}}, (s_{n}^{(2)})_{n\in\mathbb{N}} \subseteq L^1(X,\mu) \) such that

\[ \lim_{n \to \infty} \|f^+ - s_n^{(1)}\|_{L^1(X,\mu)} = 0, \quad \lim_{n \to \infty} \|f^- - s_n^{(2)}\|_{L^1(X,\mu)} = 0. \]

This implies by triangle inequality that

\[ \lim_{n \to \infty} \|f - (s_n^{(1)} - s_n^{(2)})\|_{L^1(X,\mu)} = \lim_{n \to \infty} \|f^+ - f^+ - (s_n^{(1)} - s_n^{(2)})\|_{L^1(X,\mu)} \]

\[ \leq \lim_{n \to \infty} \left( \|f^+ - s_n^{(1)}\|_{L^1(X,\mu)} + \|s_n^{(2)} - f^-\|_{L^1(X,\mu)} \right) \]

\[ = \lim_{n \to \infty} \|f^+ - s_n^{(1)}\|_{L^1(X,\mu)} + \lim_{n \to \infty} \|s_n^{(2)} - f^-\|_{L^1(X,\mu)} \]

\[ = 0 + 0 = 0, \]
i.e. the function \( f \) can be approximated by stepfunctions.

Step 03.: Stepfunctions can be approximated by continuous functions with compact support. Let \( A \subseteq X \) be a measurable subset of \( X \) with \( \mu(A) < \infty \) and set \( f = \chi_A \) on \( X \). By a conclusion of the lecture we know that

\[
\mu(A) = \sup_{K \subseteq A, \ K \text{ compact}} \mu(K).
\]

Let be \( \varepsilon > 0 \). So we can choose some compact subset \( K \subseteq A \) such that

\[
\mu(K) > \mu(A) - \varepsilon,
\]

i.e.

\[
\mu(A \setminus K) = \mu(A) - \mu(K) < \mu(A) - (\mu(A) - \varepsilon) = \varepsilon.
\]

Define the sequence

\[
f_n : \rightarrow [0,1], \ x \mapsto f_n(x) := \left(1 - \frac{\text{dist}(x,K)}{n}\right)^+ := \sup \left\{0,1 - \frac{\text{dist}(x,K)}{n}\right\} \text{ for } n \in \mathbb{N}.
\]

Then we have \( f_n \in C^0(X) \) with support

\[
\text{supp}(f_n) = \{x \in X: \text{dist}(x,K) \leq n\} \text{ for } n \in \mathbb{N}.
\]

For \( n \in \mathbb{N} \) the function \( f_n \) has compact support, since for \( \delta > 0 \) we find finite many point \( x_1, \ldots, x_k \in K, k \in \mathbb{N} \), with

\[
K \subseteq \bigcup_{i=1}^{k} B_{\delta}(x_i),
\]

because the subset \( K \) is compact. Then we get that

\[
\text{supp}(f_n) \subseteq \bigcup_{i=1}^{k} B_{\delta+n}(x_i) = \bigcup_{i=1}^{k} B_{\delta+n}(x_i),
\]

and the union \( \bigcup_{i=1}^{k} B_{\delta+n}(x_i) \) is compact as a finite union of compact subsets, because of the \( \sigma \)-compactness the closed balls \( B_{\delta+n}(x_i) \) for \( i = 1, \ldots, k \) are compact. Since the support of \( f_n \) is a closed subset in a compact set we know that \( \text{supp}(f_n) \) is compact, i.e. \( f_n \in C^0(X) \). We have \( f_n = f \) on \( K \) for all \( n \in \mathbb{N} \), then it follows:

\[
\|f_n - f\|_{L^1(X, \mu)} = \int_X |f_n(x) - \chi_A(x)| \, d\mu(x) = \int_X (\chi_A(x) - f_n(x)) \, d\mu(x) \\
\leq \int_X (\chi_A(x) - \chi_K(x)) \, d\mu(x) = \int_X \chi_A \setminus K(x) \, d\mu(x) \\
= \mu(A \setminus K) < \varepsilon,
\]

i.e. \( f \) can be approximated by continuous functions with compact support.

Step 04.: Functions in \( L^1(X, \mu) \) can be approximated by continuous functions with compact support. Let \( f \in L^1(X, \mu) \) be a function. Then by step 02. there is a sequence \( (s_n)_{n \in \mathbb{N}} \subseteq L^1(X, \mu) \) of stepfunctions such that

\[
\lim_{n \to \infty} \|f - s_n\|_{L^1(X, \mu)} = 0.
\]

For every \( n \in \mathbb{N} \) by step 03. there is a sequence \( (g_k^{(n)})_{k \in \mathbb{N}} \subseteq C^0_c(X) \) of continuous functions with compact support such that

\[
\lim_{k \to \infty} \|s_n - g_k^{(n)}\|_{L^1(X, \mu)} = 0.
\]

Now we choose the diagonal sequence \( f_n := g_k^{(n)} \in C^0_c(X) \) for \( n \in \mathbb{N} \) and we have that

\[
\lim_{n \to \infty} \|f - f_n\|_{L^1(X, \mu)} = 0,
\]

i.e. the function \( f \) can be approximated by continuous function with compact support. □