Functional analysis

8. Exercise Sheet

Exercise 1 (Functions of local bounded variation)
For a function \( f \in L^1_{\text{loc}}(\mathbb{R}^d), \ d \in \mathbb{N} \), we define the functional

\[
\Phi_f : C^1_c(\mathbb{R}^d, \mathbb{R}^d) \rightarrow \mathbb{R}, \ g \mapsto -\int_{\mathbb{R}^d} f(x) \, \text{div}(g)(x) \, dx.
\]

We call the function \( f \) a function of local bounded variation if and only if for every compact subset \( K \subseteq \mathbb{R}^d \) it holds

\[
C(K) = \sup \{ \Phi_f(g) : g \in C^1_c(\mathbb{R}^d, \mathbb{R}^d) \text{ with supp}(g) \subseteq K \text{ and } |g(x)| \leq 1 \text{ on } \mathbb{R}^d \} < \infty.
\]

Show that there is a Radon measure \( \mu_f \) and a \( \mu_f \)-measurable function \( \eta_f : \mathbb{R}^d \rightarrow \mathbb{R} \) with \( |\eta_f| = 1 \) \( \mu_f \)-almost everywhere on \( \mathbb{R}^d \) such that we have

\[
\Phi_f(g) = \int_{\mathbb{R}^d} g(x) \cdot \eta_f(x) \, d\mu_f(x) \text{ for all } g \in C^0_c(\mathbb{R}^d, \mathbb{R}^d).
\]

Prove the identities

\[
\mu_f(A) = L^d(\text{div} f)(A) := \int_A |\nabla f(x)| \, dx \text{ for all } \mu_f\text{-measurable subsets } A \subseteq \mathbb{R}^d \text{ and } \eta_f = \frac{\nabla f}{|\nabla f|} \text{ for } f \in C^1(\mathbb{R}^d),
\]

Exercise 2 ((C) Weak* convergence)
Let the two functions \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) be defined by

\[
f = \sum_{k=-\infty}^{\infty} \chi_{[k,k+\frac{1}{4}]} \text{ on } \mathbb{R}, \quad g = \sum_{k=-\infty}^{\infty} \chi_{[k-\frac{1}{4},k]} \text{ on } \mathbb{R}.
\]

Show that the sequences \( f_n, g_n, h_n : (0,1) \rightarrow \mathbb{R}, \ n \in \mathbb{N} \), defined by

\[
f_n(x) = f(nx), \ g_n(x) = g(nx) \text{ and } h_n(x) = f_n(x)g_n(x) \text{ for } x \in (0,1)
\]

converge weakly* in \( L^\infty((0,1)) \) to functions \( f, g \) resp. a function \( h : (0,1) \rightarrow \mathbb{R} \), but it holds that \( h \neq fg \) on \( (0,1) \).

Exercise 3 (Weak convergence)
For every \( \varepsilon > 0 \) we define

\[
f_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}, \ f_\varepsilon(x) = \sqrt{\frac{\varepsilon}{x^2 + \varepsilon^2}}.
\]

Show that for every \( \varepsilon > 0 \) we have that \( \|f_\varepsilon\|_{L^2(\mathbb{R})} = \sqrt{\pi} \). Check the weak convergence of \( f_\varepsilon \) and \( f_\varepsilon^2 \) in \( L^2(\mathbb{R}) \) for \( \varepsilon \rightarrow 0^+ \).

Exercise 4 ((C) Weak convergence in \( L^2((-\pi, \pi)) \))

1. Show that if \( (u_n)_{n \in \mathbb{N}} \subseteq L^2((-\pi, \pi)) \) is a bounded sequence such that

\[
\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} u_n(x) \varphi(x) \, dx = 0 \text{ for all } \varphi \in C^\infty_c((-\pi, \pi)),
\]

the sequence \( (u_n)_{n \in \mathbb{N}} \) converges weakly to 0 in \( L^2((-\pi, \pi)) \) if \( n \rightarrow \infty \).

2. Check if the sequence \( u_n(x) := \sin(nx), \ n \in \mathbb{N}, \ x \in (-\pi, \pi) \), converges pointwise on \( (-\pi, \pi) \) and/or converges weakly in \( L^2((-\pi, \pi)) \) if \( n \rightarrow \infty \).