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Functional analysis

8. Exercise Sheet

Exercise 1 (Functions of local bounded variation)

For a function $f \in L^1_{\text{loc}}\left(\mathbb{R}^d\right)$, $d \in \mathbb{N}$, we define the functional

$$\Phi_f \colon C^1_c\left(\mathbb{R}^d, \mathbb{R}^d\right) \to \mathbb{R},\ g \mapsto -\int_{\mathbb{R}^d} f(x)\,\mathrm{div}(g)(x)\mathrm{d}x.$$

We call the function f a function of local bounded variation if and only if for every compact subset $K \subseteq \mathbb{R}^d$ it holds

$$C(K) = \sup\left\{\Phi_f(g) \colon g \in C^1_c\left(\mathbb{R}^d, \mathbb{R}^d\right) \text{ with } \mathrm{supp}\,(g) \subseteq K \text{ and } |g(x)| \leq 1 \text{ on } \mathbb{R}^d\right\} < \infty.$$

Show that there is a Radon measure μ_f and a μ_f-measurable function $\eta_f \colon \mathbb{R}^d \to \mathbb{R}^d$ with $|\eta_f| = 1$ μ_f-almost everywhere on $\mathbb{R}^d$ such that we have

$$\Phi_f(g) = \int_{\mathbb{R}^d} g(x) \cdot \eta_f(x)\mathrm{d}\mu_f(x) \text{ for all } g \in C^0_c\left(\mathbb{R}^d, \mathbb{R}^d\right).$$

Prove the identities

$$\mu_f(A) = \mathcal{L}^d \llcorner |\nabla f|\,(A) := \int_A |\nabla f(x)|\,\mathrm{d}x \text{ for all } \mu_f\text{-measurable subsets } A \subseteq \mathbb{R}^d \text{ and } \eta_f = \frac{\nabla f}{|\nabla f|} \text{ for } f \in C^1\left(\mathbb{R}^d\right),$$

Exercise 2 ((C) Weak* convergence)

Let the two functions $f, g \colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f = \sum_{k=-\infty}^{\infty} \chi_{\left[k, k+\frac{1}{2}\right]},\ g = \sum_{k=-\infty}^{\infty} \chi_{\left[k-\frac{1}{2}, k\right]} \text{ on } \mathbb{R}.$$

Show that the sequences $f_n, g_n, h_n \colon (0,1) \to \mathbb{R}$, $n \in \mathbb{N}$, defined by

$$f_n(x) = f(nx),\ g_n(x) = g(nx) \text{ and } h_n(x) = f_n(x)g_n(x) \text{ for } x \in (0,1)$$

converge weakly* in $L^\infty((0,1))$ to functions f, g resp. a function $h \colon (0,1) \to \mathbb{R}$, but it holds that $h \neq fg$ on $(0,1)$.

Exercise 3 (Weak convergence)

For every $\varepsilon > 0$ we define

$$f_\varepsilon \colon \mathbb{R} \to \mathbb{R},\ f_\varepsilon(x) = \sqrt{\frac{\varepsilon}{x^2 + \varepsilon^2}}.$$

Show that for every $\varepsilon > 0$ we have that $\|f_\varepsilon\|_{L^2(\mathbb{R})} = \sqrt{\pi}$. Check the weak convergence of f_ε and f_ε^2 in $L^2(\mathbb{R})$ for $\varepsilon \to 0^+$.

Exercise 4 ((C) Weak convergence in $L^2((-\pi, \pi))$)

(1) Show that if $(u_n)_{n\in\mathbb{N}} \subseteq L^2((-\pi, \pi))$ is a bounded sequence such that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} u_n(x)\varphi(x)\mathrm{d}x = 0 \text{ for all } \varphi \in C^\infty_c((-\pi, \pi)),$$

the sequence $(u_n)_{n\in\mathbb{N}}$ converges weakly to 0 in $L^2((-\pi, \pi))$ if $n \to \infty$.

(2) Check if the sequence $u_n(x) := \sin(nx)$, $n \in \mathbb{N}$, $x \in (-\pi, \pi)$, converges pointwise on $(-\pi, \pi)$ and/ or converges weakly in $L^2((-\pi, \pi))$ if $n \to \infty$.