Functional analysis

Solutions to 8. Exercise Sheet

Exercise 1 (Functions of local bounded variation)

For a function \( f \in L^1_{\text{loc}}(\mathbb{R}^d), d \in \mathbb{N} \), we define the functional

\[
\Phi_f : C^1_c(\mathbb{R}^d, \mathbb{R}^d) \to \mathbb{R}, \quad g \mapsto -\int_{\mathbb{R}^d} f(x) \text{div}(g)(x) \, dx.
\]

We call the function \( f \) a function of local bounded variation if and only if for every compact subset \( K \subseteq \mathbb{R}^d \) it holds

\[
C(K) = \sup \{ \Phi_f(g) : g \in C^1_c(\mathbb{R}^d, \mathbb{R}^d) \text{ with supp}(g) \subseteq K \text{ and } |g(x)| \leq 1 \text{ on } \mathbb{R}^d \} < \infty.
\]

Show that there is a Radon measure \( \mu_f \) and a \( \mu_f \)-measurable function \( \eta_f : \mathbb{R}^d \to [0, \infty) \) with \( |\eta_f| = 1 \) \( \mu_f \)-almost everywhere on \( \mathbb{R}^d \) such that we have

\[
\Phi_f(g) = \int_{\mathbb{R}^d} g(x) \cdot \eta_f(x) \, d\mu_f(x) \text{ for all } g \in C^1_c(\mathbb{R}^d, \mathbb{R}^d).
\]

Prove the identities

\[
\mu_f(A) = \mathcal{L}^d \{ |\nabla f| \} (A) := \int_A |\nabla f(x)| \, dx \text{ for all } \mu_f\text{-measurable subsets } A \subseteq \mathbb{R}^d \text{ and } \eta_f = \frac{\nabla f}{|\nabla f|} \text{ for } f \in C^1(\mathbb{R}^d),
\]

Solution of Exercise 1

We denote by \( BV_{\text{loc}}(\mathbb{R}^d, \mathbb{R}) \) the set of \( L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}) \)-functions of local bounded variation. Let \( f \in BV_{\text{loc}}(\mathbb{R}^d, \mathbb{R}) \) be a function of local bounded variation. The functional \( \Phi_f \) is linear on \( C^1_c(\mathbb{R}^d, \mathbb{R}^d) \), since the divergence \( \text{div} \) and the integral are linear. If \( V \subseteq \mathbb{R}^d \) is an open and bounded subset of \( \mathbb{R}^d \), then by the theorem of Heine-Borel we know that the subset \( \overline{V} \subseteq \mathbb{R}^d \) is compact. Then we know that

\[
|\Phi_f(g)| \leq C(\overline{V}) \|g\|_{L^\infty(V, \mathbb{R}^d)} \quad \text{for all } g \in C^1_c(V, \mathbb{R}^d),
\]

i.e. \( \Phi_f \) is a continuous/ bounded linear functional on \( C^1_c(V, \mathbb{R}^d) \) with \( \|\Phi_f\| = C(\overline{V}) \). Now let \( K \subseteq \mathbb{R}^d \) be a compact subset of \( \mathbb{R}^d \) and choose some open and bounded subset \( V \subseteq \mathbb{R}^d \) with \( \overline{V} \subseteq \mathbb{R}^d \). Let \( g \in C^0_b(\mathbb{R}^d, \mathbb{R}) \) be arbitrary with \( \text{supp}(g) \subseteq K \). Then by Analysis III there is a sequence \( (g_n)_{n \in \mathbb{N}} \subseteq C^1_c(V, \mathbb{R}^d) \) such that \( g_n \to g \) uniformly on \( V \) for \( n \to \infty \), i.e.

\[
\lim_{n \to \infty} \|g - g_n\|_{L^\infty(V, \mathbb{R}^d)} = 0.
\]

By linearity of \( \Phi_f \) and the estimate above we get that

\[
|\Phi_f(g_n) - \Phi_f(g_m)| = |\Phi_f(g_n - g_m)| \leq C(\overline{V}) \|g_n - g_m\|_{L^\infty(V, \mathbb{R}^d)} \to 0 \text{ for } n, m \to \infty,
\]

since \( (g_n)_{n \in \mathbb{N}} \) is also a Cauchy-sequence in \( L^\infty(V, \mathbb{R}^d) \). This means that \( (\Phi_f(g_n))_{n \in \mathbb{N}} \subseteq \mathbb{R} \) is a Cauchy-sequence in \( \mathbb{R} \) and \( (\mathbb{R}, |\cdot|) \) is complete, i.e.

\[
A := \lim_{n \to \infty} \Phi_f(g_n) \in \mathbb{R} \text{ exists in } \mathbb{R}.
\]

If \( (h_n)_{n \in \mathbb{N}} \subseteq C^1_c(V, \mathbb{R}^d) \) is another sequence such that

\[
\lim_{n \to \infty} \|g - h_n\|_{L^\infty(V, \mathbb{R}^d)} = 0,
\]

then we also get that

\[
B := \lim_{n \to \infty} \Phi_f(g_n) \in \mathbb{R} \text{ exists in } \mathbb{R}.
\]

We get by the linearity and boundedness of \( \Phi_f \) and by the convergence of \( (g_n)_{n \in \mathbb{N}} \) and \( (h_n)_{n \in \mathbb{N}} \) that

\[
|A - B| = \lim_{n \to \infty} |\Phi_f(g_n) - \Phi_f(h_n)| = \lim_{n \to \infty} |\Phi_f(g_n - h_n)|
\]
The Representation Theorem gives us now a Radon measure for all \( \mu \). Proof

Show that the sequences for each compact subset are unique. Let \( |\Phi| \leq \lambda \) and the limit is also linear.

\[
L_f : C^0_c (\mathbb{R}^d, \mathbb{R}^d) \rightarrow \mathbb{R}
\]
with

\[
\sup \{ L_f(g) : g \in C^0 (\mathbb{R}^d, \mathbb{R}^d) \text{ with } \text{supp}(g) \subseteq K \text{ and } |g| \leq 1 \text{ on } V \} \leq C (V) < \infty
\]
for each compact subset \( K \subseteq \mathbb{R}^d \) of \( \mathbb{R}^d \) and for each open and bounded subset \( V \subseteq \mathbb{R}^d \) with \( K \subseteq V \). The Riesz Representation Theorem gives us now a Radon measure \( \mu_f \) and a \( \mu_f \)-measurable function \( \eta_f : \mathbb{R}^d \rightarrow \mathbb{R}^d \) with \( |\eta_f(x)| = 1 \) for \( \mu_f \)-almost every \( x \in \mathbb{R}^d \), such that

\[
L_f (g) = \int_{\mathbb{R}^d} g(x) \cdot \eta_f(x) \, d\mu(x) \quad \text{for all } g \in C^0_c (\mathbb{R}^d, \mathbb{R}^d).
\]

If additionally the function \( f \) is continuously differentiable on \( \mathbb{R}^d \), i.e. \( f \in C^1 (\mathbb{R}^d, \mathbb{R}^d) \), then we get with Green’s formula

\[
\Phi_f(g) = -\int_{\mathbb{R}^d} f(x) \text{div}(g)(x) \, dx = \int_{\mathbb{R}^d} \nabla f(x) \cdot g(x) \, dx = \int_{\mathbb{R}^d} g(x) \cdot \frac{\nabla f}{|\nabla f|}(x) \cdot \nabla f(x) \, dx
\]
for all \( g \in C^1 (\mathbb{R}^d, \mathbb{R}^d) \). If we now extend this to \( C^0 (\mathbb{R}^d, \mathbb{R}^d) \) we get that

\[
\Phi_f(g) = \int_{\mathbb{R}^d} g(x) \cdot \eta_f(x) \, d\mu_f(x) = L(g) = \int_{\mathbb{R}^d} g(x) \cdot \frac{\nabla f}{|\nabla f|}(x) \, d\mathcal{L}_d(\nabla f) \quad \text{for all } g \in C^0 (\mathbb{R}^d, \mathbb{R}^d), \ i.e. \ \mu_f = \mathcal{L}_d(\nabla f) \text{ and } \eta_f = \frac{\nabla f}{|\nabla f|} \mu_f \text{-almost everywhere on } \mathbb{R}^d.
\]

Exercise 2 ((C) Weak* convergence)

Let the two functions \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) be defined by

\[
f = \sum_{k=-\infty}^{\infty} \chi_{[k, k+\frac{1}{2}]}, \quad g = \sum_{k=-\infty}^{\infty} \chi_{[k-\frac{1}{2}, k]} \text{ on } \mathbb{R}.
\]

Show that the sequences \( f_n, g_n, h_n : (0, 1) \rightarrow \mathbb{R}, n \in \mathbb{N} \), defined by

\[
f_n(x) = f(nx), \quad g_n(x) = g(nx) \quad \text{and} \quad h_n(x) = f_n(x)g_n(x) \quad \text{for } x \in (0, 1)
\]
converge weakly* in \( L^\infty((0,1)) \) to functions \( f, g \) resp. a function \( h : (0, 1) \rightarrow \mathbb{R} \), but it holds that \( h \neq fg \) on \( (0,1) \).

Solution of Exercise 2

First we will show a much more general result:

Theorem: Let \( I \subseteq \mathbb{R} \) be an open and bounded interval on \( \mathbb{R} \) and let \( 1 < p \leq \infty \) and \( 1 \leq q < \infty \) be the conjugate Hölder exponent of \( p \), i.e. \( \frac{1}{p} + \frac{1}{q} = 1 \). If \( \varphi \in L^\infty(\mathbb{R}) \) is a periodic function with period \( \lambda > 0 \), i.e. \( \varphi(x + \lambda) = \varphi(x) \) for almost every \( x \in \mathbb{R} \), then the function sequence \( \varphi_n(x) := \varphi(nx), \ n \in \mathbb{N}, \ x \in I, \ ) \text{ converge weakly* in } L^p(I) \text{ to } \frac{1}{\lambda} \int_0^\lambda \varphi(x) \, dx \text{ as } n \rightarrow \infty.
\]

Proof: Set \( c := \frac{1}{\lambda} \int_0^\lambda \varphi(x) \, dx \). Since \( \varphi \in L^\infty(\mathbb{R}) \) we get that the function

\[
h : \mathbb{R} \rightarrow \mathbb{R}, \ x \mapsto \int_0^x (\varphi(\tau) - c) \, d\tau
\]
is (Lipschitz-)continuous on \( \mathbb{R} \), because we have

\[
|h(x) - h(y)| = \left| \int_0^x (\varphi(\tau) - c) \, d\tau - \int_0^y (\varphi(\tau) - c) \, d\tau \right| = \left| \int_y^x \varphi(\tau) \, d\tau + c(y - x) \right|
\]
Then there is some
By the previous result we know that \((f)\)
The functions
This implies now that if
The function \(h\) is also bounded on \(\mathbb{R}\), since by continuity it is clear that \(h\) is bounded on every compact interval on \(\mathbb{R}\) and we get for all \(k \in \mathbb{N}_0\):
\[
h(k\lambda) = \int_0^{k\lambda} (\varphi(\tau) - c) \, d\tau = \int_0^{k\lambda} \varphi(\tau) \, d\tau - k\lambda c = \sum_{l=1}^k \int_{(l-1)\lambda}^{l\lambda} (\varphi(\tau) - \lambda c) \, d\tau = \sum_{l=1}^k \left[ \int_0^{\lambda} (\varphi(\tau) - \lambda c) \, d\tau \right] = \sum_{l=1}^k \left[ \int_0^{\lambda} (\varphi(\tau) - \lambda c) \, d\tau \right] = \sum_{l=1}^k 0 = 0,
\]
and \(h(-k\lambda) = 0\) by the same calculation, i.e. \(L := \sup_{x \in \mathbb{R}} |h(x)| = \sup_{x \in [0,\lambda]} |h(x)| < \infty\), i.e. \(h\) is bounded on \(\mathbb{R}\). If \(a, b \in \mathbb{R}\) and \(J := (a, b) \subseteq I\) or \(J := [a, b] \subseteq I\) is an arbitrary interval we get that
\[
\left| \int_I (\varphi_n(x) - c) \chi_J(x) \, dx \right| = \left| \int_a^b (\varphi(nx) - c) \, dx \right| = \frac{1}{n} \left| \int_{na}^{nb} (\varphi(x) - c) \, dx \right| = \frac{1}{n} |h(nb) - h(na)| \leq 2L \to 0 \text{ as } n \to \infty.
\]
This implies now that if \(\psi : \mathbb{R} \to \mathbb{R}\) is a stepfunction we have that
\[
\lim_{n \to \infty} \int_I (\varphi_n(x) - c) \psi(x) \, dx = 0
\]
and we know that
\[
\|\varphi_n\|_{L^\infty(I)} = \|\varphi(n\cdot)\|_{L^\infty(I)} \leq \|\varphi\|_{L^\infty(\mathbb{R})} < \infty
\]
for all \(n \in \mathbb{N}\), i.e. \(\sup_{n \in \mathbb{N}} \|\varphi_n\|_{L^\infty(I)} \leq \|\varphi\|_{L^\infty(\mathbb{R})} < \infty\). Then we get by Exercise 4 (2') that the sequence \((\varphi_n)_{n \in \mathbb{N}}\) converge weakly to \(c\) in \(L^p(I)\) as \(n \to \infty\), i.e. \((\varphi_n)_{n \in \mathbb{N}}\) converge weakly* to \(c\) in \(L^p(I)'\) as \(n \to \infty\).

The functions \(f\) and \(g\) are obviously bounded by 1 and are periodic with period \(\lambda = 1 > 0\), since for \(x \in \mathbb{R}\) we have two cases: Case 01.: \(f(x) = 1\) resp. \(g(x) = 1\).
Then there is some \(k_0 \in \mathbb{Z}\) resp. \(l_0 \in \mathbb{Z}\) with \(x \in [k_0, k_0 + \frac{1}{2}]\) resp. \(x \in [l_0 - \frac{1}{2}, l_0]\), and we get that \(x + 1 \in [k_0 + 1, k_0 + 1 + \frac{1}{2}]\) resp. \(x + 1 \in [l_0 + 1 - \frac{1}{2}, l_0 + 1]\), i.e.
\[
f(x + 1) = 1 = f(x), \text{ resp. } g(x + 1) = 1 = g(x).
\]
Case 02.: \(f(x) = 0\) resp. \(g(x) = 0\).
Then for all \(k, l \in \mathbb{Z}\) we have that \(x \notin [k, k + \frac{1}{2}] \text{ resp. } x \notin [l - \frac{1}{2}, l]\), and we get that \(x + 1 \notin [k + 1, k + 1 + \frac{1}{2}] \text{ resp. } x + 1 \notin [l + 1 - \frac{1}{2}, l + 1]\) for all \(k, l \in \mathbb{Z}\), i.e.
\[
f(x + 1) = 0 = f(x), \text{ resp. } g(x + 1) = 0 = g(x).
\]
And we calculate
\[
\frac{1}{2} \int_0^1 f(x) \, dx = \int_0^1 \sum_{k=-\infty}^{\infty} \chi_{[k,k+\frac{1}{2}]}(x) \, dx = \int_0^1 \chi_{[0,\frac{1}{2}]}(x) \, dx = \int_0^{\frac{1}{2}} 1 \, dx = \frac{1}{2} - 0 = \frac{1}{2},
\]
\[
\frac{1}{2} \int_0^1 g(x) \, dx = \int_0^1 \sum_{k=-\infty}^{\infty} \chi_{[k,k+\frac{1}{2}]}(x) \, dx = \int_0^1 \chi_{[\frac{1}{2},1]}(x) \, dx = \int_0^{\frac{1}{2}} 1 \, dx = \frac{1}{2} - 1 = -\frac{1}{2}.
\]
By the previous result we know that \((f_n)_{n \in \mathbb{N}}\) and \((g_n)_{n \in \mathbb{N}}\) converge weakly* to \(\frac{1}{2}\) in \(L^\infty((0,1))\) for \(n \to \infty\). We define the function \(h : \mathbb{R} \to \mathbb{R}\) by \(h = fg\) on \(\mathbb{R}\), then it is \(h_n = h(n\cdot)\). The function \(h\) is obviously bounded by 1 and is periodic
with period $\lambda = 1 > 0$, since we have $h(x + 1) = f(x + 1)g(x + 1) = f(x)g(x) = h(x)$ for all $x \in \mathbb{R}$ by the periodicity of $f$ and $g$. For the integral we get
\[
\frac{1}{1} \int_0^1 h(x) \, dx = \int_0^1 f(x) g(x) \, dx = \int_0^1 \left( \sum_{k=-\infty}^{\infty} \chi_{[k,k+\frac{1}{2}]}(x) \right) \cdot \left( \sum_{k=-\infty}^{\infty} \chi_{[k-\frac{1}{2},k]}(x) \right) \, dx
\]
\[
= \int_0^1 \chi_{[0,\frac{1}{2}]}(x) \cdot \chi_{[1-\frac{1}{2},1]}(x) \, dx + \int_0^1 \chi_{[\frac{1}{2},1]}(x) \cdot \chi_{[\frac{1}{2},1]}(x) \, dx = \int_0^1 \chi_{\{\frac{1}{2}\}}(x) \, dx = 0,
\]
since $\{\frac{1}{2}\}$ has measure zero. By the previous result we get that the sequence $(h_n)_{n \in \mathbb{N}}$ converge weakly* to 0 in $L^\infty((0,1))$, but $0 \neq \frac{1}{2} \cdot \frac{1}{2}$.

Exercise 3  (Weak convergence)

For every $\varepsilon > 0$ we define
\[
f_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}, \quad f_\varepsilon(x) = \sqrt{\frac{\varepsilon}{x^2 + \varepsilon^2}}.
\]
Show that for every $\varepsilon > 0$ we have that $\|f_\varepsilon\|_{L^2(\mathbb{R})} = \sqrt{\pi}$. Check the weak convergence of $f_\varepsilon$ and $f_\varepsilon^2$ in $L^2(\mathbb{R})$ for $\varepsilon \rightarrow 0^+$.

Exercise 4  ((C) Weak convergence in $L^2((-\pi,\pi))$)

(1) Show that if $(u_n)_{n \in \mathbb{N}} \subseteq L^2((-\pi,\pi))$ is a bounded sequence such that
\[
\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} u_n(x) \varphi(x) \, dx = 0 \quad \text{for all } \varphi \in C_c^\infty((-\pi,\pi)),
\]
the sequence $(u_n)_{n \in \mathbb{N}}$ converges weakly to 0 in $L^2((-\pi,\pi))$ if $n \rightarrow \infty$.

(2) Check if the sequence $u_n(x) := \sin(nx), \ n \in \mathbb{N}, \ x \in (-\pi,\pi)$, converges pointwise on $(-\pi,\pi)$ and/ or converges weakly in $L^2((-\pi,\pi))$ if $n \rightarrow \infty$. 
