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Functional analysis

9. Exercise Sheet

Exercise 1 (Strong converging sequence of convex combinations)

Let $(X, \|\cdot\|_X)$ be a Banach space and let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a weakly converging sequence in X with limit $x \in X$. Show that there is a sequence $(y_n)_{n\in\mathbb{N}}$ of (finite) convex combinations of the $(x_n)_{n\in\mathbb{N}}$ which converges strongly to x for $n \to \infty$.

Exercise 2 (Dirac sequence)

Let $\varphi \in L^1(\mathbb{R}^d)$, $d \in \mathbb{N}$, be a function. Show that the Dirac sequence $\varphi_\varepsilon(x) := \varepsilon^{-d}\varphi(\varepsilon^{-1}x)$, $\varepsilon > 0$, for $x \in \mathbb{R}^d$, converges weakly* in $C_c^0(\mathbb{R}^d)'$, up to a constant, to the Dirac measure for $\varepsilon \to 0^+$.

Exercise 3 ((C) Adjoint operators in Hilbert spaces)

Let $(X, \langle\cdot,\cdot\rangle_X)$, $(Y, \langle\cdot,\cdot\rangle_Y)$ be two Hilbert spaces and let $T \in L(X,Y)$. Show that

$$\|T^*\| = \|T\|,$$

where T^* is the adjoint operator of T.

Exercise 4 ((C) The Heisenberg Uncertainty principle)

Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space and $A, B\colon H \to H$ be two linear self-adjoint operators, i.e. $A = A^*$ resp. $B = B^*$. In Quantum mechanics the different measurement parameters (also called Oberservables) can be assumed to be such linear self-adjoint operators A in some Hilbert space H. The elements $\psi \in H$ in H with $\|\psi\|_H = 1$ represent the different conditions of a particle, and we set

$$\langle A\rangle := \langle A\psi, \psi\rangle_H \ \text{(expectation value)}$$
$$\Delta A := \left\langle (A - \langle A\rangle \operatorname{Id}_H)^2 \psi, \psi \right\rangle_H^{\frac{1}{2}} \ \text{(Uncertainty)}.$$

Show that

$$(\Delta A)\cdot(\Delta B) \geq \frac{1}{2}\left|\langle [A,B]\rangle\right|,$$

where the commutator of A and B is defined by

$$[A,B] := AB - BA.$$