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Functional analysis

Solutions to 9. Exercise Sheet

Exercise 1 (Strong converging sequence of convex combinations)

Let $(X, \|\cdot\|_X)$ be a Banach space and let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a weakly converging sequence in X with limit $x \in X$. Show that there is a sequence $(y_n)_{n\in\mathbb{N}}$ of (finite) convex combinations of the $(x_n)_{n\in\mathbb{N}}$ which converges strongly to x for $n \to \infty$.

Solution of Exercise 1

Remark: (The closure of convex sets is convex) If A is a convex set, then also its closure $\overline{A}$ is convex.
This is true, since for arbitrary $x, y \in \overline{A}$ and $\lambda \in [0,1]$ we find two sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq A$ such that $x_n \to x$ and $y_n \to y$ in A as $n \to \infty$, and we know by convexity of A that $\lambda x_n + (1-\lambda)y_n \in A$ for every $n \in \mathbb{N}$. Then we can write:

$$\lambda x + (1-\lambda)y = \lim_{n\to\infty} (\lambda x_n + (1-\lambda)y_n) \in \overline{A},$$

i.e. the $\overline{A}$ is convex. □
Remark: (The other implication is not true) If A is a set and the closure $\overline{A}$ is convex, then the set A is not convex in general.
The counterexample is quite simple, just choose $A := [0,1] \setminus \left\{\frac{1}{2}\right\} \subseteq \mathbb{R}$ with closure $\overline{A} = [0,1] \subseteq \mathbb{R}$. It is obvious that $\overline{A}$ is convex, but A is not convex because $0, 1 \in A$ and $\lambda = \frac{1}{2} \in [0,1]$, but

$$\frac{1}{2} \cdot 0 + \left(1 - \frac{1}{2}\right) \cdot 1 = 0 + \frac{1}{2} = \frac{1}{2} \notin [0,1] \setminus \left\{\frac{1}{2}\right\} = A.$$

□

We define the subset

$$C := \operatorname{conv}\{x_n : n \in \mathbb{N}\} \subseteq X.$$

Then the closure $\overline{C}$ of C is a closed subset of X and since the subset C is convex (as the convex hull) we know that also the closure $\overline{C}$ is convex (see Remark above). Now we assume that $x \notin \overline{C}$. Then by the geometric version of the Hahn-Banach Theorem there is a hyperplane H which separates $\{x\}$ and C, i.e. we can find a functional $\Phi \in X'$ and some $\delta \in \mathbb{R}$ such that

$$\Phi(x) < \delta < \Phi(y) \text{ for all } y \in C.$$

Then it follows:

$$0 < \delta - \Phi(x) < \Phi(x_n) - \Phi(x) \text{ for all } n \in \mathbb{N}.$$

By the weak convergence we get a contradiction, since we have for $n \to \infty$

$$0 < \delta - \Phi(x) \leq 0.$$

This means that our assumption was wrong, i.e. $x \in \overline{C}$. □

Exercise 2 (Dirac sequence)

Let $\varphi \in L^1(\mathbb{R}^d)$, $d \in \mathbb{N}$, be a function. Show that the Dirac sequence $\varphi_\varepsilon(x) := \varepsilon^{-d}\varphi(\varepsilon^{-1}x)$, $\varepsilon > 0$, for $x \in \mathbb{R}^d$, converges weakly* in $C_c^0(\mathbb{R}^d)'$, up to a constant, to the Dirac measure for $\varepsilon \to 0^+$.

Solution of Exercise 2

Remark: ($L^1(X,\mu)$ in $C_c^0(X)'$) For a function $\psi \in L^1(X)$ we define the functional

$$\Phi_\psi : C_c^0(X) \to \mathbb{R},\ f \mapsto \int_X f(x)\psi(x)\,\mathrm{d}\mu(x).$$

The functional Φ_ψ is well-defined by the Hölder inequality, since we have

$$\int_X |f(x)\psi(x)| \,\mathrm{d}\mu(x) \leq \|\psi\|_{L^1(X,\mu)} \|f\|_{L^\infty(X,\mu)} = \|\psi\|_{L^1(X,\mu)} \|f\|_{C^0(X)} < \infty \text{ for all } f \in C_c^0(X).$$

And the functional Φ_ψ is linear by the linearity of the product and the integral. And we have boundedness/ continuity with

$$|\Phi_\psi(f)| = \left| \int_X \psi(x) f(x) \mathrm{d}\mu(x) \right| \leq \int_X |\psi(x)f(x)| \,\mathrm{d}\mu(x) \leq \|\psi\|_{L^1(X,\mu)} \|f\|_{C^0(X)} \text{ for all } f \in C_c^0(X),$$

i.e. $\|\Phi_\psi\| \leq \|\psi\|_{L^1(X,\mu)}$, this means that $\Phi_\psi \in C_c^0(X)'$.
Now we define the map

$$\Phi\colon L^1(X,\mu) \to C_c^0(X)',\ \varphi \mapsto \Phi_\varphi,$$

then Φ is well-defined as we could see above and it's bounded by 1, i.e. $\|\Phi\| \leq 1$. From the proof of the Riesz Representation Theorem we can see that Φ is also an isometrie, i.e.

$$\|\Phi_\psi\| = \|\psi\|_{L^1(X,\mu)} \text{ for all } \psi \in L^1(X,\mu).$$

Then Φ is especially injective, and we can say that $L^1(X,\mu) \subseteq C_c^0(X)'$ in the sense that if $\psi \in L^1(X,\mu)$, then the functional

$$\Phi_\psi \in C_c^0(X)'.$$

Let $f \in C_c^0\left(\mathbb{R}^d\right)$ be an arbitrary continuous function with compact support. First we consider that $\varphi(\cdot) f(\varepsilon \cdot)$ is a (Lebesgue-)measurable function on $\mathbb{R}^d$ for every $\varepsilon > 0$. Then we have pointwise convergence everywhere on $\mathbb{R}^d$, since we get for $x \in \mathbb{R}^d$ by continuity of f:

$$\varphi(x) f(\varepsilon x) \to \varphi(x) f(0) \text{ as } \varepsilon \to 0^+,$$

and $f(0)\varphi(\cdot)$ is also (Lebesgue-)measurable on $\mathbb{R}^d$. Since the function f is bounded we have for $x \in \mathbb{R}^d$

$$0 \leq |\varphi(x) f(\varepsilon x)| \leq \sup_{y\in\mathbb{R}^d} |f(\varepsilon y)| \cdot |\varphi(x)| = \sup_{y\in\mathbb{R}^d} |f(y)| \cdot |\varphi(x)| = \|f\|_{L^\infty(\mathbb{R}^d)} |\varphi(x)|$$

with $\|f\|_{L^\infty(\mathbb{R}^d)} |\varphi(\cdot)| \in L^1\left(\mathbb{R}^d\right)$, because we have

$$\int_{\mathbb{R}^d} \left| \|f\|_{L^\infty(\mathbb{R}^d)} |\varphi(x)| \right| \mathrm{d}x = \|f\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} |\varphi(x)| \,\mathrm{d}x = \|f\|_{L^\infty(\mathbb{R}^d)} \|\varphi\|_{L^1(\mathbb{R}^d)} < \infty,$$

Then we get by the transformation formula:

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi_\varepsilon(x) f(x) \mathrm{d}x &= \int_{\mathbb{R}^d} \varepsilon^{-d} \varphi\left(\frac{x}{\varepsilon}\right) f(x) \mathrm{d}x \\ &= \int_{\mathbb{R}^d} \varphi(y) f(\varepsilon y) \,\mathrm{d}y \end{aligned}$$

for all $\varepsilon > 0$. And now with the Theorem of Lebesgue (Dominated convergence) we get that

$$\lim_{\varepsilon\to 0+} \int_{\mathbb{R}^d} \varphi_\varepsilon(x) f(x) \mathrm{d}x = \int_{\mathbb{R}^d} \lim_{\varepsilon\to 0+} \varphi(x) f(\varepsilon x) \mathrm{d}x = \int_{\mathbb{R}^d} \varphi(x) f(0) \mathrm{d}x = \int_{\mathbb{R}^d} \varphi(x) \mathrm{d}x f(0) = \left(\int_{\mathbb{R}^d} \varphi(x) \mathrm{d}x \right) \delta_0(f).$$

This shows that

$$\varphi_\varepsilon \rightharpoonup^* \left(\int_{\mathbb{R}^d} \varphi(x) \mathrm{d}x \right) \delta_0 \text{ in } C_c^0\left(\mathbb{R}^d\right)' \text{ as } \varepsilon \to 0^+.$$

□

Exercise 3 ((C) Adjoint operators in Hilbert spaces)

Let $(X, \langle\cdot,\cdot\rangle_X)$, $(Y, \langle\cdot,\cdot\rangle_Y)$ be two Hilbert spaces and let $T \in L(X,Y)$. Show that

$$\|T^*\| = \|T\|,$$

where T^* is the adjoint operator of T.

Solution of Exercise 3

By the definition of the norm and with the Cauchy-Schwarz inequality we get that

$$\|T\|^2 = \sup_{x\in X\colon \|x\|_X \leq 1} \|Tx\|_Y^2 = \sup_{x\in X\colon \|x\|_X \leq 1} \langle Tx, Tx\rangle_Y = \sup_{x\in X\colon \|x\|_X \leq 1} \langle x, T^*Tx\rangle_X$$

$$
\begin{aligned}
&\leq \sup_{x\in X:\, \|x\|_X\leq 1} \left(\|x\|_X \cdot \|T^*Tx\|_X\right) \leq \|T^*\| \sup_{x\in X:\, \|x\|_X\leq 1} \|Tx\|_Y = \|T^*\|\,\|T\|\,, \\
\|T^*\|^2 &= \sup_{y\in Y:\, \|y\|_Y\leq 1} \|T^*y\|_X^2 = \sup_{y\in Y:\, \|y\|_Y\leq 1} \langle T^*y, T^*y\rangle_X = \sup_{y\in Y:\, \|y\|_Y\leq 1} \langle TT^*y, y\rangle_Y \\
&\leq \sup_{y\in Y:\, \|y\|_Y\leq 1} \left(\|y\|_Y \cdot \|TT^*y\|_Y\right) \leq \|T\| \sup_{y\in Y:\, \|y\|_Y\leq 1} \|T^*y\|_X = \|T\|\,\|T^*\|\,,
\end{aligned}
$$

i.e. if $\|T\| = 0$, then also $\|T^*\| = 0$ (and $T = 0$ on X resp. $T^* = 0$ on Y) and otherwise we get

$$
\|T\| \leq \|T^*\| \leq \|T\|\,, \text{ i.e. } \|T\| = \|T^*\|\,.
$$

□

Remark: $(T = T^{**})$ You also can argue that

$$
\|T^*\| \leq \|T\| = \|T^{**}\| = \left\|(T^*)^*\right\| \leq \|T^*\|\,,
$$

since we have for arbitrary $x \in X$ and $y \in Y$ that

$$
\begin{aligned}
\langle y, T^{**}x\rangle_Y &= \langle y, (T^*)^*\, x\rangle_Y = \langle T^*y, x\rangle_X \\
&= \overline{\langle x, T^*y\rangle_X} = \overline{\langle Tx, y\rangle_Y} \\
&= \langle y, Tx\rangle_Y \\
\Leftrightarrow\;\; 0 &= \langle y, T^{**}x\rangle_Y - \langle y, Tx\rangle_Y = \langle y, T^{**}x - Tx\rangle_Y = \langle y, (T^{**} - T)\, x\rangle_Y\,,
\end{aligned}
$$

i.e. $(T^{**} - T)\, x = 0$ for all $x \in X$ and this implies that $T^{**}x = Tx$ for all $x \in X$, in other words $T^{**} = T$ on X. □

Exercise 4 ((C) The Heisenberg Uncertainty principle)

Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space and $A, B\colon H \to H$ be two linear self-adjoint operators, i.e. $A = A^*$ resp. $B = B^*$. In Quantum mechanics the different measurement parameters (also called Oberservables) can be assumed to be such linear self-adjoint operators A in some Hilbert space H. The elements $\psi \in H$ in H with $\|\psi\|_H = 1$ represent the different conditions of a particle, and we set

$$
\begin{aligned}
\langle A\rangle &:= \langle A\psi, \psi\rangle_H \text{ (expectation value)} \\
\Delta A &:= \left\langle \left(A - \langle A\rangle \operatorname{Id}_H\right)^2 \psi, \psi \right\rangle_H^{\frac{1}{2}} \text{ (Uncertainty).}
\end{aligned}
$$

Show that

$$
(\Delta A)\cdot(\Delta B) \geq \frac{1}{2}\left|\langle [A, B]\rangle\right|,
$$

where the commutator of A and B is defined by

$$
[A, B] := AB - BA.
$$

Solution of Exercise 4

We define

$$
f := \left(A - \langle A\rangle \operatorname{Id}_H\right)\psi,\;\; g\left(B - \langle B\rangle \operatorname{Id}_H\right)\psi \text{ in } H.
$$

And since the operators A, B and Id_H are self-adjoint operators we have that

$$
\begin{aligned}
\langle A\rangle &= \langle A\psi, \psi\rangle = \langle \psi, A^*\psi\rangle = \langle \psi, A\psi\rangle = \overline{\langle A\psi, \psi\rangle} = \overline{\langle A\rangle}, \\
\langle B\rangle &= \langle B\psi, \psi\rangle = \langle \psi, B^*\psi\rangle = \langle \psi, B\psi\rangle = \overline{\langle B\psi, \psi\rangle} = \overline{\langle B\rangle}, \\
&\text{i.e. } \langle A\rangle\,, \langle B\rangle \in \mathbb{R}, \\
\left(A - \langle A\rangle \operatorname{Id}_H\right)^* &= A^* - \left(\langle A\rangle \operatorname{Id}_H\right)^* = A^* - \overline{\langle A\rangle}\operatorname{Id}_H^* = A^* - \langle A\rangle \operatorname{Id}_H, \\
\left(B - \langle B\rangle \operatorname{Id}_H\right)^* &= B^* - \left(\langle B\rangle \operatorname{Id}_H\right)^* = B^* - \overline{\langle B\rangle}\operatorname{Id}_H^* = B^* - \langle B\rangle \operatorname{Id}_H,
\end{aligned}
$$

i.e. $A - \langle A\rangle \operatorname{Id}_H$ and $B - \langle B\rangle \operatorname{Id}_H$ are also self-adjoint operators. We calulate for the uncertainty of A resp. B:

$$
\begin{aligned}
(\Delta A)^2 &= \left\langle \left(A - \langle A\rangle \operatorname{Id}_H\right)^2 \psi, \psi\right\rangle_H = \left\langle \left(A - \langle A\rangle \operatorname{Id}_H\right)\left(A - \langle A\rangle \operatorname{Id}_H\right)\psi, \psi\right\rangle_H \\
&= \left\langle \left(A - \langle A\rangle \operatorname{Id}_H\right)\psi, \left(A - \langle A\rangle \operatorname{Id}_H\right)^* \psi\right\rangle_H \\
&= \left\langle \left(A - \langle A\rangle \operatorname{Id}_H\right)\psi, \left(A - \langle A\rangle \operatorname{Id}_H\right)\psi\right\rangle_H = \langle f, f\rangle_H\,,
\end{aligned}
$$

$$
\begin{aligned}
(\Delta B)^2 &= \left\langle (B - \langle B\rangle \mathrm{Id}_H)^2 \psi, \psi \right\rangle_H = \langle (B - \langle B\rangle \mathrm{Id}_H)(B - \langle B\rangle \mathrm{Id}_H)\psi, \psi\rangle_H \\
&= \left\langle (B - \langle B\rangle \mathrm{Id}_H)\psi, (B - \langle B\rangle \mathrm{Id}_H)^* \psi\right\rangle_H \\
&= \langle (B - \langle B\rangle \mathrm{Id}_H)\psi, (B - \langle B\rangle \mathrm{Id}_H)\psi\rangle_H = \langle g, g\rangle_H ,
\end{aligned}
$$

i.e. $0 \le \Delta A = \sqrt{\langle f, f\rangle_H}$ resp. $0 \le \Delta B = \sqrt{\langle g, g\rangle_H}$ By the Cauchy-Schwarz inequality we get that

$$
|\langle f, g\rangle_H|^2 \le \|f\|_H^2 \|g\|_H^2 = \langle f, f\rangle_H \langle g, g\rangle_H = (\Delta A \cdot \Delta B)^2 .
$$

By Analysis I we know that for any $z \in \mathbb{C}$ we have

$$
|z|^2 \ge |\mathrm{Im}(z)|^2 = \left| \frac{z - \overline{z}}{2\mathrm{i}} \right|^2 .
$$

Now we get for $z = \langle f, g\rangle_H$:

$$
|\langle f, g\rangle_H|^2 \ge \left| \frac{\langle f, g\rangle_H - \overline{\langle f, g\rangle_H}}{2\mathrm{i}} \right|^2 = \left| \frac{\langle f, g\rangle_H - \langle g, f\rangle_H}{2\mathrm{i}} \right|^2 .
$$

Next we look at $\langle f, g\rangle_H$:

$$
\begin{aligned}
\langle f, g\rangle_H &= \langle (A - \langle A\rangle \mathrm{Id}_H)\psi, (B - \langle B\rangle \mathrm{Id}_H)\psi\rangle_H \\
&= \langle A\psi - \langle A\rangle \psi, B\psi - \langle B\rangle \psi\rangle_H \\
&= \langle A\psi, B\psi\rangle_H - \langle B\rangle \langle A\psi, \psi\rangle_H - \langle A\rangle \langle \psi, B\psi\rangle_H + \langle A\rangle \langle B\rangle \langle \psi, \psi\rangle_H \\
&= \langle A\psi, B\psi\rangle_H - \langle B\rangle \langle A\rangle - \langle A\rangle \langle B\rangle + \langle A\rangle \langle B\rangle \|\psi\|_H^2 \\
&= \langle A\psi, B\psi\rangle_H - 2\langle A\rangle \langle B\rangle + \langle A\rangle \langle B\rangle \|\psi\|_H^2 .
\end{aligned}
$$

Then we look at $\langle g, f\rangle_H$:

$$
\begin{aligned}
\langle g, f\rangle_H &= \langle (B - \langle B\rangle \mathrm{Id}_H)\psi, (A - \langle A\rangle \mathrm{Id}_H)\psi\rangle_H \\
&= \langle B\psi - \langle B\rangle \psi, A\psi - \langle A\rangle \psi\rangle_H \\
&= \langle B\psi, A\psi\rangle_H - \langle A\rangle \langle B\psi, \psi\rangle_H - \langle B\rangle \langle \psi, B\psi\rangle_H + \langle B\rangle \langle A\rangle \langle \psi, \psi\rangle_H \\
&= \langle B\psi, A\psi\rangle_H - \langle A\rangle \langle B\rangle - \langle B\rangle \langle A\rangle + \langle B\rangle \langle A\rangle \|\psi\|_H^2 \\
&= \langle B\psi, A\psi\rangle_H - 2\langle A\rangle \langle B\rangle + \langle A\rangle \langle B\rangle \|\psi\|_H^2 .
\end{aligned}
$$

Now by subtraction we get that

$$
\begin{aligned}
\langle f, g\rangle_H - \langle g, f\rangle_H &= \langle A\psi, B\psi\rangle_H - \langle B\psi, A\psi\rangle_H \\
&= \langle A\psi, B^*\psi\rangle_H - \langle B\psi, A^*\psi\rangle_H \\
&= \langle BA\psi, \psi\rangle_H - \langle AB\psi, \psi\rangle_H \\
&= \langle BA\psi - AB\psi, \psi\rangle_H = \langle (BA - AB)\psi, \psi\rangle_H \\
&= \langle -[A, B]\psi, \psi\rangle_H = -\langle [A, B]\rangle .
\end{aligned}
$$

This implies now:

$$
|\langle f, g\rangle_H|^2 \ge \left| \frac{\langle f, g\rangle_H - \langle g, f\rangle_H}{2\mathrm{i}} \right|^2 = \left| \frac{-\langle [A, B]\rangle}{2\mathrm{i}} \right|^2 = \frac{1}{4} |\langle [A, B]\rangle|^2 .
$$

In the end we have

$$
\Delta A \cdot \Delta B = \sqrt{\langle f, f\rangle_H} \cdot \sqrt{\langle g, g\rangle_H} \ge \sqrt{|\langle f, g\rangle_H|} \ge \frac{1}{2} |\langle [A, B]\rangle| .
$$

□