Functional analysis

Solutions to 9. Exercise Sheet

Exercise 1  (Strong converging sequence of convex combinations)

Let \((X, \| \cdot \|_X)\) be a Banach space and let \((x_n)_{n \in \mathbb{N}} \subseteq X\) be a weakly converging sequence in \(X\) with limit \(x \in X\). Show that there is a sequence \((y_n)_{n \in \mathbb{N}}\) of (finite) convex combinations of the \((x_n)_{n \in \mathbb{N}}\) which converges strongly to \(x\) for \(n \to \infty\).

Solution of Exercise 1

Remark: (The closure of convex sets is convex) If \(A\) is a convex set, then also its closure \(\overline{A}\) is convex. This is true, since for arbitrary \(x, y \in A\) and \(\lambda \in [0, 1]\) we find two sequences \((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subseteq A\) such that \(x_n \to x\) and \(y_n \to y\) in \(A\) as \(n \to \infty\), and we know by convexity of \(A\) that \(\lambda x_n + (1 - \lambda) y_n \in A\) for every \(n \in \mathbb{N}\). Then we can write:

\[
\lambda x + (1 - \lambda)y = \lim_{n \to \infty} (\lambda x_n + (1 - \lambda) y_n) \in A,
\]

i.e. the \(\overline{A}\) is convex. \(\square\)

Remark: (The other implication is not true) If \(A\) is a set and the closure \(\overline{A}\) is convex, then the set \(A\) is not convex in general. The counterexample is quite simple, just choose \(A := [0, 1] \setminus \{1/2\} \subseteq \mathbb{R}\) with closure \(\overline{A} = [0, 1] \subseteq \mathbb{R}\). It is obvious that \(\overline{A}\) is convex, but \(A\) is not convex because \(0, 1 \in A\) and \(\lambda = 1/2 \in [0, 1]\), but

\[
\frac{1}{2} \cdot 0 + \left(1 - \frac{1}{2}\right) \cdot 1 = 0 + \frac{1}{2} = \frac{1}{2} \notin [0, 1] \setminus \{1/2\} = A.
\]

We define the subset

\[
C := \text{conv}\{x_n : n \in \mathbb{N}\} \subseteq X.
\]

Then the closure \(\overline{C}\) of \(C\) is a closed subset of \(X\) and since the subset \(C\) is convex (as the convex hull) we know that also the closure \(\overline{C}\) is convex (see Remark above). Now we assume that \(x \notin \overline{C}\). Then by the geometric version of the Hahn-Banach Theorem there is a hyperplane \(H\) which separates \(\{x\} \) and \(C\), i.e. we can find a functional \(\Phi \in X'\) and some \(\delta \in \mathbb{R}\) such that

\[
\Phi(x) < \delta < \Phi(y) \quad \text{for all } y \in C.
\]

Then it follows:

\[
0 < \delta - \Phi(x) < \Phi(x_n) - \Phi(x) \quad \text{for all } n \in \mathbb{N}.
\]

By the weak convergence we get a contradiction, since we have for \(n \to \infty\)

\[
0 < \delta - \Phi(x) \leq 0.
\]

This means that our assumption was wrong, i.e. \(x \in \overline{C}\). \(\square\)

Exercise 2  (Dirac sequence)

Let \(\varphi \in L^1(\mathbb{R}^d), \ d \in \mathbb{N}\), be a function. Show that the Dirac sequence \(\varphi_\varepsilon(x) := \varepsilon^{-d} \varphi(\varepsilon^{-1} x), \varepsilon > 0\), for \(x \in \mathbb{R}^d\), converges weakly* in \(C_0^0(\mathbb{R}^d)\), up to a constant, to the Dirac measure for \(\varepsilon \to 0^+\).

Solution of Exercise 2

Remark: \((L^1(X, \mu) \cap C_0^0(X))'\) For a function \(\psi \in L^1(X)\) we define the functional

\[
\Phi_\psi : C_0^0(X) \to \mathbb{R}, \ f \mapsto \int_X f(x) \psi(x) d\mu(x).
\]
The functional $\Phi_\psi$ is well-defined by the Hölder inequality, since we have
\[
\int_X |f(x)\psi(x)| \, d\mu(x) \leq \|\psi\|_{L^1(X,\mu)} \|f\|_{L^\infty(X,\mu)} = \|\psi\|_{L^1(X,\mu)} \|f\|_{C^0(X)} < \infty \quad \text{for all } f \in C^0_c(X).
\]
And the functional $\Phi_\psi$ is linear by the linearity of the product and the integral. And we have boundedness/continuity with
\[
|\Phi_\psi(f)| = \left| \int_X \psi(x)f(x) \, d\mu(x) \right| \leq \int_X |\psi(x)f(x)| \, d\mu(x) \leq \|\psi\|_{L^1(X,\mu)} \|f\|_{C^0(X)} \quad \text{for all } f \in C^0_c(X),
\]
i.e. $\|\Phi_\psi\| \leq \|\psi\|_{L^1(X,\mu)}$, this means that $\Phi_\psi \in C^0_c(X)'$.

Now we define the map
\[
\Phi: L^1(X,\mu) \to C^0_c(X)', \quad \varphi \mapsto \Phi_\varphi,
\]
then $\Phi$ is well-defined as we could see above and it’s bounded by 1, i.e. $\|\Phi\| \leq 1$. From the proof of the Riesz Representation Theorem we can see that $\Phi$ is also an isometric, i.e.
\[
\|\Phi_\psi\| = \|\psi\|_{L^1(X,\mu)} \quad \text{for all } \psi \in L^1(X,\mu).
\]
Then $\Phi$ is especially injective, and we can say that $L^1(X,\mu) \subseteq C^0_c(X)'$ in the sense that if $\psi \in L^1(X,\mu)$, then the functional $\Phi_\psi \in C^0_c(X)'$.

Let $f \in C^0_c(\mathbb{R}^d)$ be an arbitrary continuous function with compact support. First we consider that $\varphi(\cdot)f(\cdot)$ is a (Lebesgue-)measurable function on $\mathbb{R}^d$ for every $\varepsilon > 0$. Then we have pointwise convergence everywhere on $\mathbb{R}^d$, since we get for $x \in \mathbb{R}^d$ by continuity of $f$:
\[
\varphi(x)f(\varepsilon x) \to \varphi(x)f(0) \quad \text{as } \varepsilon \to 0^+,
\]
and $f(0)\varphi(\cdot)$ is also (Lebesgue-)measurable on $\mathbb{R}^d$. Since the function $f$ is bounded we have for $x \in \mathbb{R}^d$
\[
0 \leq |\varphi(x)f(\varepsilon x)| \leq \sup_{y \in \mathbb{R}^d} |f(\varepsilon y)| \cdot |\varphi(x)| = \sup_{y \in \mathbb{R}^d} |f(y)| \cdot |\varphi(x)| = \|f\|_{L^\infty(\mathbb{R}^d)} |\varphi(x)|
\]
with $\|f\|_{L^\infty(\mathbb{R}^d)} |\varphi(\cdot)| \in L^1(\mathbb{R}^d)$, because we have
\[
\int_{\mathbb{R}^d} \|f\|_{L^\infty(\mathbb{R}^d)} |\varphi(x)| \, dx = \|f\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} |\varphi(x)| \, dx = \|f\|_{L^\infty(\mathbb{R}^d)} \|\varphi\|_{L^1(\mathbb{R}^d)} < \infty,
\]
Then we get by the transformation formula:
\[
\int_{\mathbb{R}^d} \varphi(x)f(x) \, dx = \int_{\mathbb{R}^d} \varepsilon^{-d} \varphi \left( \frac{x}{\varepsilon} \right) f(x) \, dx = \int_{\mathbb{R}^d} \varphi(y)f(\varepsilon y) \, dy
\]
for all $\varepsilon > 0$. And now with the Theorem of Lebesgue (Dominated convergence) we get that
\[
\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d} \varphi(x)f(x) \, dx = \int_{\mathbb{R}^d} \lim_{\varepsilon \to 0^+} \varphi(x)f(\varepsilon x) \, dx = \int_{\mathbb{R}^d} \varphi(x)f(0) \, dx = \int_{\mathbb{R}^d} \varphi(x)dxf(0) = \left( \int_{\mathbb{R}^d} \varphi(x) \, dx \right) \delta_0(f).
\]
This shows that
\[
\varphi_\varepsilon \rightarrow^* \left( \int_{\mathbb{R}^d} \varphi(x) \, dx \right) \delta_0 \in C^0_c(\mathbb{R}^d)' \quad \text{as } \varepsilon \to 0^+.
\]

Exercise 3 (C) Adjoint operators in Hilbert spaces

Let $(X, \langle \cdot, \cdot \rangle_X)$, $(Y, \langle \cdot, \cdot \rangle_Y)$ be two Hilbert spaces and let $T \in L(X,Y)$. Show that
\[
\|T^*\| = \|T\|
\]
where $T^*$ is the adjoint operator of $T$.

Solution of Exercise 3

By the definition of the norm and with the Cauchy-Schwarz inequality we get that
\[
\|T\|^2 = \sup_{x \in X: \|x\|_X \leq 1} \|Tx\|_Y^2 = \sup_{x \in X: \|x\|_X \leq 1} \langle Tx, Tx \rangle_Y = \sup_{x \in X: \|x\|_X \leq 1} \langle x, T^*Tx \rangle_X
\]
We define $A$, i.e. $(A, \psi) = \langle A \psi, \psi \rangle$, where the commutator of $A$ and $B$ is defined by $[A, B] := AB - BA$.

Remark: $(T = T^*)$ You also can argue that

$$
\|T^*\| \leq \|T\| = \|T^*\| = \|T\|,
$$

since we have for arbitrary $x \in X$ and $y \in Y$ that

$$
\langle y, T^* x \rangle_y = \langle y, (T^*)^* x \rangle_y = \langle (T^*)^* y, x \rangle_x = \langle x, T y \rangle_x = \langle y, T x \rangle_y
$$

\[
\Leftrightarrow 0 = \langle y, T^* x \rangle_y - \langle y, T x \rangle_y = \langle y, T^{**} x - T x \rangle_y = \langle y, (T^{**} - T) x \rangle_y,
\]

i.e. $(T^{**} - T) x = 0$ for all $x \in X$ and this implies that $T^{**} x = T x$ for all $x \in X$, in other words $T^{**} = T$ on $X$.

Exercise 4 ((C) The Heisenberg Uncertainty principle)

Let $(H, \langle \cdot, \cdot \rangle_H)$ be a Hilbert space and $A, B : H \to H$ be two linear self-adjoint operators, i.e. $A = A^*$ resp. $B = B^*$. In Quantum mechanics the different measurement parameters (also called Observables) can be assumed to be such linear self-adjoint operators $A$ in some Hilbert space $H$. The elements $\psi \in H$ in $H$ with $\|\psi\|_H = 1$ represent the different conditions of a particle, and we set

$$
(A) := \langle A \psi, \psi \rangle_H \text{ (expectation value)}
$$

$$
\Delta A := \langle (A - \langle A \rangle H)^2 \psi, \psi \rangle_H^\frac{1}{2} \text{ (Uncertainty)}.
$$

Show that

$$
(\Delta A) \cdot (\Delta B) \geq \frac{1}{2} |\langle [A, B] \rangle|,
$$

where the commutator of $A$ and $B$ is defined by

$$
[A, B] := AB - BA.
$$

Solution of Exercise 4

We define

$$
f := (A - \langle A \rangle H) \psi, \ g := (B - \langle B \rangle H) \psi \text{ in } H.
$$

And since the operators $A$, $B$ and $\text{Id}_H$ are self-adjoint operators we have that

$$
\langle A \rangle = \langle A \psi, \psi \rangle = \langle \psi, A^* \psi \rangle = \langle \psi, A \psi \rangle = \langle A \psi, \psi \rangle = \langle A \rangle,
$$

$$
\langle B \rangle = \langle B \psi, \psi \rangle = \langle \psi, B^* \psi \rangle = \langle \psi, B \psi \rangle = \langle B \psi, \psi \rangle = \langle B \rangle,
$$

i.e. $\langle A \rangle, \langle B \rangle \in \mathbb{R}$,

$$
(A - \langle A \rangle H^*)^* = A^* - \langle A \rangle H^* = A^* - \langle A \rangle H_H,
$$

$$
(B - \langle B \rangle H^*)^* = B^* - \langle B \rangle H^* = B^* - \langle B \rangle H_H,
$$

i.e. $A - \langle A \rangle H_H$ and $B - \langle B \rangle H_H$ are also self-adjoint operators. We calculate for the uncertainty of $A$ resp. $B$:

$$
(\Delta A)^2 = \langle (A - \langle A \rangle H^2 \psi, \psi \rangle_H = \langle (A - \langle A \rangle H) (A - \langle A \rangle H) \psi, \psi \rangle_H
$$

$$
= \langle (A - \langle A \rangle H) \psi, (A - \langle A \rangle H)^* \psi \rangle_H
$$

$$
= \langle (A - \langle A \rangle H) \psi, (A - \langle A \rangle H) \psi \rangle_H = \langle f, f \rangle_H,
$$

3
\[(\Delta B)^2 = \langle (B - (B) \text{Id}_H)^2, \psi, \psi \rangle_H = \langle (B - (B) \text{Id}_H) (B - (B) \text{Id}_H), \psi, \psi \rangle_H = \langle (B - (B) \text{Id}_H) \psi, (B - (B) \text{Id}_H)^* \psi \rangle_H = \langle (B - (B) \text{Id}_H) \psi, (B - (B) \text{Id}_H)^* \psi \rangle_H = \langle g, g \rangle_H, \]
i.e. \(0 \leq \Delta A = \sqrt{\langle f, f \rangle_H} \) resp. \(0 \leq \Delta B = \sqrt{\langle g, g \rangle_H} \). By the Cauchy-Schwarz inequality we get that
\[|\langle f, g \rangle_H|^2 \leq \|f\|_H^2 \|g\|_H^2 = \langle f, f \rangle_H \langle g, g \rangle_H = (\Delta A \cdot \Delta B)^2.\]

By Analysis I we know that for any \(z \in \mathbb{C}\) we have
\[|z|^2 \geq |\text{Im}(z)|^2 = \left|\frac{z - \bar{z}}{2i}\right|^2.\]

Now we get for \(z = \langle f, g \rangle_H\):
\[|\langle f, g \rangle_H|^2 \geq \left|\frac{\langle f, g \rangle_H - \langle f, f \rangle_H}{2i}\right|^2 = \left|\frac{\langle f, g \rangle_H - \langle g, f \rangle_H}{2i}\right|^2.\]

Next we look at \(\langle f, g \rangle_H\):
\[\langle f, g \rangle_H = \langle (A - (A) \text{Id}_H) \psi, (B - (B) \text{Id}_H) \psi \rangle_H = \langle A\psi - (A) \psi, B\psi - (B) \psi \rangle_H = \langle A\psi, B\psi \rangle_H - (B) \langle A\psi, \psi \rangle_H - (A) \langle B\psi, \psi \rangle_H + \langle A \rangle \langle B \rangle \langle \psi, \psi \rangle_H = 1.\]

Then we look at \(\langle g, f \rangle_H\):
\[\langle g, f \rangle_H = \langle (B - (B) \text{Id}_H) \psi, (A - (A) \text{Id}_H) \psi \rangle_H = \langle B\psi - (B) \psi, A\psi - (A) \psi \rangle_H = \langle B\psi, A\psi \rangle_H - (A) \langle B\psi, \psi \rangle_H - (B) \langle A\psi, \psi \rangle_H + \langle A \rangle \langle B \rangle \langle \psi, \psi \rangle_H = 1.\]

Now by subtraction we get that
\[\langle f, g \rangle_H - \langle g, f \rangle_H = \langle A\psi, B\psi \rangle_H - (B\psi, A\psi) H = \langle A\psi, B^* \psi \rangle_H - (B\psi, A^* \psi) H = \langle B A\psi, \psi \rangle_H - (B A^* \psi, \psi) H = \langle B (A - AB) \psi, \psi \rangle_H = -\langle [A, B] \psi, \psi \rangle_H.\]

This implies now:
\[|\langle f, g \rangle_H|^2 \geq \left|\frac{\langle f, g \rangle_H - \langle g, f \rangle_H}{2i}\right|^2 = \left|\frac{\langle [A, B] \psi, \psi \rangle_H}{2i}\right|^2 = \frac{1}{4} |\langle [A, B] \psi, \psi \rangle_H|^2.\]

In the end we have
\[\Delta A \cdot \Delta B = \sqrt{\langle f, f \rangle_H} \cdot \sqrt{\langle g, g \rangle_H} \geq \frac{1}{2} |\langle [A, B] \psi, \psi \rangle_H|^2.\]