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Functional analysis

Solutions to 10. Exercise Sheet

Exercise 1 (Sobolev spaces are reflexive and seperable)

Show that the Sobolev space $W^{1,p}(\Omega)$ is seperable for $1 \leqslant p < \infty$ and is reflexive for $p \in (1, \infty)$.
(Hint: Use the map $J\colon W^{1,p}(\Omega) \to X_p(\Omega),\ u \mapsto (u, \nabla u)$, where $X_p(\Omega) := \bigtimes_{i=1}^{d+1} L^p(\Omega)$.)

Solution of Exercise 1

Lemma 1: If X is a reflexive Banach space and $A \subseteq X$ a closed subspace of X, then also A is reflexive.
Let $a'' \in A''$ be arbitrary. We have to show that there is some element $a \in A$ such that

$$a''(a') = a'(a) \text{ for all } a' \in A'.$$

If $T\colon X \to X''$ is the canonnical embedding, and define $\Phi\colon X' \to X'',\ x' \mapsto a''(x'|_A)$, this is well-defined, since

$$\left|\Phi\left((x')\right)\right| = \left|a''\left(x'|_A\right)\right| \leqslant \|a''\| \cdot \|x'|_A\| \leqslant \|a''\| \|x'\|.$$

Since X is reflexive, the map T is surjective and we find a $x \in X$ such that

$$x'(x) = \Phi\left(x'\right) \text{ for all } x' \in X'.$$

If x would not be in A, then by the Hahn-Banach-Theorem (since A is closed) we find some linear bounded functional $f \in X'$ such that $f|_A = 0$ and $f(x) = 1$, but this is a contradiction to

$$0 = \Phi(f) = f(x) = 1,$$

i.e. $x \in A$ and A is also reflexive. □

Lemma 2: Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be two Banach spaces and $A \in L(X, Y)$ be bijective. Then also the dual operator $A'\colon Y' \to X'$ is bijective. Is additionally the operator A an isometrie, then the dual operator A' is also an isometrie and the space X is reflexive if and only if Y is reflexive.

Exercise 2 ((C) Sobolev spaces in Hölder spaces)

Show that for the real unit interval $I = (0, 1) \subseteq \mathbb{R}$ the Sobolev space $W^{1,p}(I)$, $p \in [1, \infty]$, is continuously embedded in the Hölder space $C^{0,1-\frac{1}{p}}(I)$.

Solution of Exercise 2

Since the set I has finite Lebesgue measure (equal to 1) we know that $L^p(I) \subseteq L^q(I)$ for all $p, q \in [1, \infty]$ with $p \geqslant q$, i.e. $W^{1,p}(I) \subseteq W^{1,1}(I)$. Therefore we have for $f \in W^{1,p}(I)$, where we choose the continuous representative on $\overline{I}$:

$$f(x_2) - f(x_1) = \int_{x_1}^{x_2} f'(x)\mathrm{d}x \text{ for all } x_1, x_2 \in I.$$

It follows with Hölder inequality for all $x_1, x_2 \in [0, 1]$:

$$\begin{aligned} |f(x_2) - f(x_1)| &= \left|\int_{x_1}^{x_2} f'(x)\mathrm{d}x\right| \leqslant \int_{x_1}^{x_2} 1 \cdot |f'(x)|\,\mathrm{d}x \\ &\leqslant \left(\int_{x_1}^{x_2} 1^{\frac{1}{1-\frac{1}{p}}}\mathrm{d}x\right)^{1-\frac{1}{p}} \cdot \left(\int_{x_1}^{x_2} 1 \cdot |f'(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \end{aligned}$$

$$= \left(\int_{x_1}^{x_2} 1 \mathrm{d}x\right)^{1-\frac{1}{p}} \cdot \|f'\|_{L^p(I)}$$
$$= |x_1 - x_2|^{1-\frac{1}{p}} \|f'\|_{L^p(I)},$$

i.e.

$$[f]_{1-\frac{1}{p}, I} \leqslant \|f'\|_{L^p(I)}.$$

Let $x_0 \in [0,1]$ be arbitrary, then we have for all $y \in [0,1]$:

$$\begin{aligned}|f(y)| &= |f(y) - f(x_0) + f(x_0)| \leqslant |f(y) - f(x_0)| + |f(x_0)| \\ &\leqslant |y - x_0|^{1-\frac{1}{p}} \|f'\|_{L^p(I)} + |f(x_0)| \\ &\leqslant 1^{1-\frac{1}{p}} \|f'\|_{L^p(I)} + |f(x_0)| \\ &= \|f'\|_{L^p(I)} + |f(x_0)|,\end{aligned}$$

i.e.

$$\|f\|_{C^0(I)} \leqslant |f(x_0)| + \|f'\|_{L^p(I)}.$$

Then we have

$$\|f\|_{C^{0,1-\frac{1}{p}}(I)} = \|f\|_{C^0(I)} + [f]_{1-\frac{1}{p}, I} \leqslant |f(x_0)| + \|f'\|_{L^p(I)} + \|f'\|_{L^p(I)} = |f(x_0)| + 2\|f'\|_{L^p(I)}.$$

Now choose $x_0 \in [0,1]$ such that

$$\min_{x \in I} |f(x)| = |f(x_0)|,$$

since f is continuous on $[0,1]$ and $[0,1]$ is compact. Then we have

$$|f(x_0)| = \int_0^1 |f(x_0)| \,\mathrm{d}x \leqslant \int_0^1 |f(x)| \mathrm{d}x = \|f\|_{L^1(I)} \leqslant \|\overline{1}\|_{L^{p'}(I)} \cdot \|f\|_{L^p(I)} = \|f\|_{L^p(I)},$$

where $p' \in [1,\infty]$ the conjugate Hölder exponent, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$, and $\overline{1}$ the constant 1 function with

$$\|\overline{1}\|_{L^{p'}(I)} = 1.$$

This implies now

$$\|f\|_{C^{0,1-\frac{1}{p}}(I)} \leqslant |f(x_0)| + 2\|f'\|_{L^p(I)} \leqslant \|f\|_{L^p(I)} + 2\|f'\|_{L^p(I)} = \|f\|_{W^{1,p}(I)} + \|f'\|_{L^p(I)} \leqslant 2\|f\|_{W^{1,p}(I)},$$

i.e. the Sobolevspace $W^{1,p}(I)$ is continuously embedded in the Hölder space $C^{0,1-\frac{1}{p}}(I)$. □

Exercise 3 ($D^2u = 0$ implies affin linearity)

Let $u \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^d)$ be a Sobolev function with $D^2u = 0$ on $\mathbb{R}^d$. Show that the function u is affin linear, i.e. there are two constants $a \in \mathbb{R}$ and $b \in \mathbb{R}^d$ such that

$$u(x) = a + \langle b, x\rangle \text{ for all } x \in \mathbb{R}^d.$$

Exercise 4 ((C) Chain rule for weak derivatives)

Let $u \in L^1_{\mathrm{loc}}(\Omega)$ be a function with $Du \in L^1_{\mathrm{loc}}(\Omega)$. Show that for the absolute value of u we have $|u| \in W^{1,1}_{\mathrm{loc}}(\Omega)$ with

$$D(|u|) = \operatorname{sign}(u)\, Du \text{ on } \Omega.$$

(Hint: Approximate the absolute value function by $f_\varepsilon(z) = \sqrt{z^2 + \varepsilon^2}$, $z \in \mathbb{R}$, for $\varepsilon > 0$.)

Solution of Exercise 4

It is $|u| \in L^1_{\mathrm{loc}}(\Omega)$ and $f_\varepsilon \in C^1(\Omega)$, i.e. $f_\varepsilon \in W^{1,1}_{\mathrm{loc}}(\Omega)$ for all $\varepsilon > 0$ and we get

$$\int_K |f_\varepsilon(|u(x)|)| \,\mathrm{d}x = \int_K \left|\sqrt{|u(x)|^2 + \varepsilon^2}\right| \mathrm{d}x \leqslant \int_K \left|\sqrt{2\max\{|u(x)|, \varepsilon\}^2}\right| \mathrm{d}x$$

$$= \sqrt{2}\int_K \max\{|u(x)|, \varepsilon\}\,\mathrm{d}x = \sqrt{2}\left(\int_K |u(x)|\mathrm{d}x + \int_K \varepsilon\mathrm{d}x\right) < \infty$$

for all compact subsets $K \subseteq \Omega$ and all $\varepsilon > 0$, i.e. $f_\varepsilon(|u|) \in L^1_{\mathrm{loc}}(\Omega)$. Then we have for the derivative by chain rule:

$$\begin{aligned}\nabla f_\varepsilon(|u|)(x) &= \nabla\sqrt{u^2+\varepsilon^2}(x) = \frac{2u(x)}{2\sqrt{u^2(x)+\varepsilon^2}} \cdot \nabla u(x) \\ &= \frac{u(x)}{\sqrt{u^2(x)+\varepsilon^2}}\nabla u(x) \to \mathrm{sign}(u(x))\nabla u(x) \in L^1_{\mathrm{loc}}(\Omega) \text{ as } \varepsilon \to 0^+\end{aligned}$$

for all $x \in \Omega$. By the theorem of Lebesgue we get that

$$\begin{aligned}\int_\Omega \mathrm{sign}(u(x))\nabla u(x)\varphi(x)\mathrm{d}x &= \lim_{\varepsilon\to 0^+}\int_\Omega \nabla f_\varepsilon(|u(x)|)\cdot\varphi(x)\mathrm{d}x \\ &= -\lim_{\varepsilon\to 0^+}\int_\Omega f_\varepsilon(|u(x)|)\cdot\nabla\varphi(x)\mathrm{d}x \\ &= -\lim_{\varepsilon\to 0^+} |u(x)|\cdot\nabla\varphi(x)\mathrm{d}x\end{aligned}$$

for all $\varphi \in C_c^\infty(\Omega)$, i.e. $|u| \in W^{1,1}_{\mathrm{loc}}(\Omega)$ with $D(|u|) = \mathrm{sign}(u)\nabla u$. □