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Functional analysis

Solutions to 11. Exercise Sheet

Remark: (Corollary of the Hahn-Banach Theorem) Let $(X, \|\cdot\|_X)$ be a normed space. Then it is

$$\|x\|_X = \sup\{f(x) \colon f \in X' \text{ with } \|f\| \leq 1\} \text{ for all } x \in X.$$

Exercise 1 ((C) Projection into the quotient space)

Let $(X, \|\cdot\|_X)$ be a Banach space and $V \subseteq X$ be a closed subspace of X. We equip the quotient space X/V with the norm

$$\|[x]\|_{X/V} := \inf_{v \in V} \|x + v\|_X \,,$$

then $\left(X/V, \|\cdot\|_{X/V}\right)$ is a Banach space. Show that the projection $P \colon X \to X/V,\ x \mapsto [x]$ is an open map with norm $\|P\| \leq 1$. Conclude that $\|P\| = 1$ if and only if $V \neq X$, otherwise it is zero.

Solution of Exercise 1

By the definition of the $\|\cdot\|_{X/V}$ (and since $0 \in V$) we get that

$$\|Px\|_{X/V} = \|[x]\|_{X/V} = \inf_{v \in V} \|x + v\|_X \leq \|x + 0\|_X = \|x\|_X \text{ for all } x \in X,$$

i.e. $\|P\| \leq 1$.
If $V \neq X$ we can choose some element $x \in X \backslash V$, i.e. $\|[x]\|_{X/V} > 0$. For every $\varepsilon > 0$ we find an element $v_\varepsilon \in V$ such that

$$\|x + v_\varepsilon\|_X < \|[x]\|_{X/V} + \varepsilon.$$

It follows

$$\begin{aligned} \|P\| \, \|x + v_\varepsilon\|_X \geq \|P(x + v_\varepsilon)\|_{X/V} &= \|[x + v_\varepsilon]\|_{X/V} = \|[x]\|_{X/V} \\ &> \|x + v_\varepsilon\|_X - \varepsilon \\ \Leftrightarrow \ (\|P\| - 1) \|x + v_\varepsilon\|_X &> -\varepsilon \\ \Leftrightarrow \ \varepsilon > (1 - \|P\|) \|x + v_\varepsilon\|_X &\geq 0, \end{aligned}$$

i.e. $\|P\| = 1$ as $\varepsilon \to 0^+$.
If $V = X$ then $P = 0$ and we have $\|P\| = 0$. □

Exercise 2 (Weak formulation of the Neumann problem)

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with C^1-boundary and outer unit normal ν, the coefficients $a_{ij} \in C^1\left(\overline{\Omega}\right),\ b_i, c, f \in C^0\left(\overline{\Omega}\right)$. Show that $u \in C^2\left(\overline{\Omega}\right)$ solves the Neumann-problem

$$\begin{cases} -\sum_{i=1}^d \partial_i \left(\sum_{j=1}^d a_{ij} \partial_j u\right) + \sum_{i=1}^d b_i \partial_i u + cu = f & \text{in } \Omega \\ \sum_{i=1}^d \nu_i \sum_{j=1}^d a_{ij} \partial_j u = 0 & \text{on } \partial\Omega \end{cases},$$

if and only if for all $\varphi \in C^\infty\left(\overline{\Omega}\right)$ it holds

$$\int_\Omega \left[\sum_{i=1}^d \partial_i \varphi(x) \sum_{j=1}^d a_{ij}(x) \partial_j u(x) + \varphi(x) \left(\sum_{i=1}^d b_i(x) \partial_i u(x) + c(x) u(x)\right)\right] \mathrm{d}x = \int_\Omega \varphi(x) f(x) \mathrm{d}x.$$

Solution of Exercise 2

"$\Rightarrow$": Let $u \in C^2\left(\overline{\Omega}\right)$ be a solution of the Neumann-problem and $\varphi \in C^\infty\left(\overline{\Omega}\right)$, then we get with the Gauß Theorem:

$$\begin{aligned}
\int_\Omega \varphi(x) f(x) \mathrm{d}x &= \int_\Omega \varphi(x) \left[-\sum_{i=1}^d \partial_i \left(\sum_{j=1}^d a_{ij} \partial_j u \right)(x) + \sum_{i=1}^d b_i(x) \partial_i u(x) + c(x) u(x) \right] \mathrm{d}x \\
&= -\sum_{i=1}^d \int_\Omega \varphi(x) \partial_i \left(\sum_{j=1}^d a_{ij} \partial_j u \right)(x) \mathrm{d}x + \int_\Omega \left[\sum_{i=1}^d b_i(x) \partial_i u(x) + c(x) u(x) \right] \mathrm{d}x \\
&= \sum_{i=1}^d \int_\Omega \partial_i \varphi(x) \left(\sum_{j=1}^d a_{ij} \partial_j u \right)(x) \mathrm{d}x - \sum_{i=1}^d \int_{\partial\Omega} \nu_i(x) \varphi \left(\sum_{j=1}^d a_{ij} \partial_j u \right)(x) \mathrm{d}x \\
&\quad + \int_\Omega \left[\sum_{i=1}^d b_i(x) \partial_i u(x) + c(x) u(x) \right] \mathrm{d}x \\
&= \int_\Omega \left[\sum_{i=1}^d \partial_i \varphi(x) \sum_{j=1}^d a_{ij}(x) \partial_j u(x) + \varphi(x) \left(\sum_{i=1}^d b_i(x) \partial_i u(x) + c(x) u(x) \right) \right] \mathrm{d}x.
\end{aligned}$$

"$\Leftarrow$": Let $u \in C^2\left(\overline{\Omega}\right)$ with the property that for all $\varphi \in C^\infty\left(\overline{\Omega}\right)$ we have

$$\int_\Omega \left[\sum_{i=1}^d \partial_i \varphi(x) \sum_{j=1}^d a_{ij}(x) \partial_j u(x) + \varphi(x) \left(\sum_{i=1}^d b_i(x) \partial_i u(x) + c(x) u(x) \right) \right] \mathrm{d}x = \int_\Omega \varphi(x) f(x) \mathrm{d}x.$$

Let $\varphi_1 \in C_c^\infty(\Omega) \subseteq C^\infty\left(\overline{\Omega}\right)$ be arbitrary. We get with integration by parts:

$$\begin{aligned}
\int_\Omega \varphi_1(x) f(x) \mathrm{d}x &= \int_\Omega \left[\sum_{i=1}^d \partial_i \varphi_1(x) \sum_{j=1}^d a_{ij}(x) \partial_j u(x) + \varphi_1(x) \left(\sum_{i=1}^d b_i(x) \partial_i u(x) + c(x) u(x) \right) \right] \mathrm{d}x \\
&= \int_\Omega \varphi_1(x) \left[\sum_{i=1}^d \partial_i \left(\sum_{j=1}^d a_{ij} \partial_j u \right)(x) + \sum_{i=1}^d b_i(x) \partial_i u(x) + c(x) u(x) \right] \mathrm{d}x,
\end{aligned}$$

i.e.

$$\sum_{i=1}^d \partial_i \left(\sum_{j=1}^d a_{ij} \partial_j u \right)(x) + \sum_{i=1}^d b_i(x) \partial_i u(x) + c(x) u(x) = f(x) \text{ for all } x \in \Omega.$$

Let $\varphi_2 \in C^\infty\left(\overline{\Omega}\right)$ be arbitrary, then we get with the Gauß Theorem

$$\begin{aligned}
\int_\Omega \varphi_2(x) f(x) \mathrm{d}x &= \int_\Omega \left[\sum_{i=1}^d \partial_i \varphi_2(x) \sum_{j=1}^d a_{ij}(x) \partial_j u(x) + \varphi_2(x) \left(\sum_{i=1}^d b_i(x) \partial_i u(x) + c(x) u(x) \right) \right] \mathrm{d}x \\
&= \int_\Omega \varphi_2(x) \left[\sum_{i=1}^d \partial_i \left(\sum_{j=1}^d a_{ij} \partial_j u \right)(x) + \sum_{i=1}^d b_i(x) \partial_i u(x) + c(x) u(x) \right] \mathrm{d}x \\
&\quad + \int_{\partial\Omega} \varphi_2(x) \sum_{i=1}^d \nu_i(x) \sum_{j=1}^d a_{ij}(x) \partial_j u(x) \mathrm{d}x \\
&= \int_\Omega \varphi_2(x) f(x) \mathrm{d}x + \int_\Omega \varphi_2(x) \sum_{i=1}^d \nu_i(x) \sum_{j=1}^d a_{ij}(x) \partial_j u(x) \mathrm{d}x,
\end{aligned}$$

i.e.

$$0 = \int_{\partial\Omega} \varphi_2(x) \sum_{i=1}^d \nu_i(x) \sum_{j=1}^d a_{ij}(x) \partial_j u(x) \mathrm{d}x.$$

Since the function $\varphi_2 \in C^\infty\left(\overline{\Omega}\right)$ was arbitrary it follows that

$$\sum_{i=1}^d \nu_i \sum_{j=1}^d a_{ij}(x) = 0 \text{ for all } x \in \partial\Omega.$$

This implies now that $u \in C^2\left(\overline{\Omega}\right)$ solves the Neumann-Problem. $\square$

Exercise 3 (Dual space of $W_0^{1,2}(\Omega)$)

Let $X = L^2(\Omega) \times L^2\left(\Omega, \mathbb{R}^d\right)$ be the product space and we choose the norms $\|\cdot\|_X$, $\|\cdot\|_{W^{1,2}(\Omega)}$ such that the embedding $J \colon W_0^{1,2}(\Omega) \to X,\ u \mapsto (u, \nabla u)$ is isometric. Define the map

$$P \colon X \to W_0^{1,2}(\Omega)',\ P(f,F)(u) = \int_\Omega \left(f(x)u(x) + \langle F(x), \nabla u(x)\rangle \right).$$

Show that:

(1) Orthogonal decomposition: $X = \ker(P) \oplus \operatorname{range}(J)$.

(2) The map P is surjective and for $\varphi \in W_0^{1,2}(\Omega)'$ we have

$$\|\varphi\| = \inf\left\{ \|(f,F)\|_X : P(f,F) = \varphi \right\}.$$

Exercise 4 ((C) Green Operator)

Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded and $a \in L^\infty\left(\Omega, M_d(\mathbb{R})\right)$ be elliptic. Consider the operator

$$G = I \circ L^{-1} \circ I' \colon L^2(\Omega) \to L^2(\Omega),$$

where

$$\begin{aligned} &L \colon W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)',\ (Lv)(u) = \int_\Omega \langle \nabla u(x), a(x)\nabla v(x)\rangle \,\mathrm{d}x, \\ &I \colon W_0^{1,2}(\Omega) \to L^2(\Omega),\ Iv = v, \\ &I' \colon L^2(\Omega) \to W_0^{1,2}(\Omega)',\ \left(I'f\right)(u) = \int_\Omega f(x)u(x)\mathrm{d}x. \end{aligned}$$

Show that $LGf = f$ for $f \in L^2(\Omega)$ and show that if a is symmetric, then the operator G is self-adjoint with respect to $\langle\cdot,\cdot\rangle_{L^2(\Omega)}$, i.e. $G = G^*$.

Solution of Exercise 4

It is $LGf = LIL^{-1}I'f = LL^{-1}I'f = I'f = f$ for all $f \in L^2(\Omega)$, where the last equality-sign is the identification of the dual elements of $L^2(\Omega)$ with $L^2(\Omega)$-functions by the Riesz Representation Theorem. Now it is for $f, g \in L^2(\Omega)$:

$$\langle Gf, g\rangle_{L^2(\Omega)} = \int_\Omega Gf(x)g(x)\mathrm{d}x = LGg(Gf) = LGf(Gg) = \langle f, Gg\rangle_{L^2(\Omega)},$$

since

$$Lu(v) = \int_\Omega \langle \nabla u(x), a(x)\nabla v(x)\rangle \,\mathrm{d}x = \int_\Omega \left\langle a(x)^T \nabla u(x), \nabla v(x)\right\rangle \mathrm{d}x = \int_\Omega \langle a(x)\nabla u(x), \nabla v(x)\rangle \,\mathrm{d}x = Lv(u)$$

for $u, v \in W_0^{1,2}(\Omega)$, i.e. $G^* = G$ and therefore G is self-adjoint. □