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Functional analysis

12. Exercise Sheet

Exercise 1 ((C) Hilbert-Schmidt-Integraloperators)

Let $I := (0,1) \subseteq \mathbb{R}$ be the open unit interval and $k \in L^2(I \times I, \mathbb{C})$ be the kernel of the operator

$$K\colon L^2(I,\mathbb{C}) \to L^2(I,\mathbb{C}),\ (Kf)(x) = \int_I k(x,y)f(y)\mathrm{d}y.$$

Show that:

(1) The operator K is well-defined and bounded with $\|K\| \leq \|k\|_{L^2(I\times I,\mathbb{C})}$.

(2) The adjoint operator K^* of K has the integralrepresentation

$$(K^*f)(x) = \int_I k^*(x,y)f(y)\mathrm{d}y \text{ for } f \in L^2(I,\mathbb{C}),\ x \in I,$$

where $k^*(x,y) := \overline{k(y,x)}$ for all $(x,y) \in I \times I$.

(3) The operator K is compact.
(Hint: Use the fact that the set of $(x,y) \mapsto \lambda\chi_{[a,b]}(x)\chi_{[\alpha,\beta]}(y)$, $\lambda \in \mathbb{C}$, is dense in $L^2(I \times I, \mathbb{C})$.)

Exercise 2 (The adjoint of the gradient)

What is the adjoint operator of the gradient

$$\nabla\colon W_0^{1,2}(\Omega) \to L^2\left(\Omega, \mathbb{R}^d\right)?$$

Exercise 3 (Some Perturbation)

Let $I = (0,\infty) \subseteq \mathbb{R}$ and $\lambda \in \mathbb{R}\backslash\{0\}$ be a constant. Define the operator $A_\lambda\colon W_0^{1,2}(I) \to L^2(I),\ u \mapsto u' + \lambda u$. Show that

(1) $\|u'\|_{L^2(I)} \leq \|A_\lambda u\|_{L^2(I)}$, $\|u\|_{L^2(I)} \leq \frac{1}{|\lambda|}\|A_\lambda u\|_{L^2(I)}$.

(2) The operator A_λ is injective and the image range$(A_\lambda) \subseteq L^2(I)$ is closed.

(3) It is $(\text{range}(A_\lambda))^\perp = \{0\}$ and $\text{ind}(A_\lambda) = 0$ for $\lambda > 0$, $(\text{range}(A_\lambda))^\perp = \text{lin}\left\{x \mapsto e^{\lambda x}\right\}$ and $\text{ind}(A_\lambda) = -1$ for $\lambda < 0$.

(4) The operator $A_0 u := u'$ (case $\lambda = 0$) is injective with dense image range$(A_0) \subseteq L^2(I)$, but the image range$(A_0) \subseteq L^2(I)$ is not closed.

Exercise 4 ((C) To the Ehrling-Lemma)

1. Let $R > 0$ be a radius and $\Omega := B_R(0) \subseteq \mathbb{R}^d$. Show that for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that for all $u \in C^2\left(\overline{\Omega}\right)$ we have

$$\|\nabla u\|_{C^0(\Omega)} \leq \varepsilon \left\|D^2 u\right\|_{C^0(\Omega)} + C_\varepsilon \|u\|_{C^0(\Omega)}$$

and give a concrete estimate for the constant C_ε.

2. Let $u \in C^2([0,1])$ be a solution of the linear differential equation

$$au'' + bu' + du = 0 \text{ in } (0,1)$$

with coefficients $a, b, d \in C^0([0,1])$ and $a \geq c_0 > 0$ on $[0,1]$, where $c_0 > 0$ is a positive constant. Then there is a constant $C = C(a,b,d) > 0$ such that

$$\|u\|_{C^2([0,1])} \leq C \|u\|_{C^0([0,1])} .$$