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Functional analysis

13. Exercise Sheet

Exercise 1 ((C) Infinite linear system)

Show that the infinite linear system

$$x_i + \sum_{j=1}^{\infty} a_{ij} x_j = b_i \text{ for } i \in \mathbb{N}$$

has for every $b \in l^2(\mathbb{N}, \mathbb{R})$ an unique solution $x \in l^2(\mathbb{N}, \mathbb{R})$, if the matrix $(a_{ij})_{1 \leq i,j \leq N}$ is for every $N \in \mathbb{N}$ positive semi-definite with $\sum\limits_{i,j=1}^{\infty} a_{ij}^2 < \infty$.

Exercise 2 (Solvability of the Neumann-Problem)

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with C^1-boundary and outer unit normal ν, $f \in L^2(\Omega)$. For functions $a \in L^\infty(\Omega, M_d(\mathbb{R}))$ and $q \in L^\infty(\Omega)$ we define the operator $L\colon W^{1,2}(\Omega) \to W^{1,2}(\Omega)'$ by

$$(Lv)(u) = \int_\Omega [\langle a(x)\nabla v(x), \nabla u(x)\rangle + q(x)v(x)u(x)]\, \mathrm{d}x \text{ for } u, v \in W^{1,2}(\Omega).$$

Additionally $a(\cdot)$ is symmetric and elliptic on Ω with some constant $\mu > 0$. Show that:

(1) The operator L is an isomorphism, if $q(x) \geq \lambda > 0$ for all $x \in \Omega$ and some $\lambda > 0$.

(2) The operator L is a Fredholm-Operator with index zero.
(Hint: Use without proof that the embedding $W^{1,2}(\Omega) \subseteq L^2(\Omega)$ is compact, if $\Omega \subseteq \mathbb{R}^d$ is a bounded domain with C^1-boundary)

(3) The condition $\int_\Omega f(x)\mathrm{d}x = 0$ is necessary and sufficient for the existence of a weak solution to the Neumann-Problem

$$\begin{cases} -\operatorname{div}(a\nabla v) = f & \text{in } \Omega \\ \sum\limits_{i=1}^{d} \nu_i \sum\limits_{j=1}^{d} a_{ij}\partial_j v = 0 & \text{on } \partial\Omega \end{cases}$$

Exercise 3 (Solvability criteria)

Consider the operator $L\colon W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)'$, $L = L_0 + K$ defined in the lecture with measurable coefficients a, b, c, q and let the operator L_0 be elliptic with constant $\mu > 0$. Show for every $\varphi \in W_0^{1,2}(\Omega)'$ the equivalence of the following statements (Theorem 10.20 in the lecture):

(1) $Lv = \varphi$ has a solution $v \in W_0^{1,2}(\Omega)$.

(2) $\varphi(u) = 0$ for all $u \in \ker(L^*)$.

Here $L^* := L' \circ J$, where $J\colon W^{1,2}(\Omega) \to W^{1,2}(\Omega)''$ is the canonical embedding. What is the meaning of (2), if the right-hand side is some L^2-function f?

Exercise 4 ((C) The spectrum of a multiplication operator)

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^d$, $X := C_b^0(\Omega) := \{f\colon \Omega \to \mathbb{K} |\ f \text{ is continuous and bounded on } \Omega\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $m \in X$ and T_m be the multiplication operator on X, i.e.

$$T_m\colon X \to X,\ f \mapsto m \cdot f.$$

What is the spectrum $\sigma(T_m)$? Determine the type of the spectral values and the resolvent $R(\lambda, T_m)$ for all $\lambda \in \rho(T_m)$.