Functional analysis

13. Exercise Sheet

Exercise 1 ((C) Infinite linear system)

Show that the infinite linear system

\[ x_i + \sum_{j=1}^{\infty} a_{ij} x_j = b_i \text{ for } i \in \mathbb{N} \]

has for every \( b \in l^2(\mathbb{N}, \mathbb{R}) \) an unique solution \( x \in l^2(\mathbb{N}, \mathbb{R}) \), if the matrix \((a_{ij})_{1 \leq i,j \leq N}\) is for every \( N \in \mathbb{N} \) positive semi-definite with \( \sum_{i,j=1}^{\infty} a_{ij}^2 < \infty \).

Exercise 2 (Solvability of the Neumann-Problem)

Let \( \Omega \subseteq \mathbb{R}^d \) be a bounded domain with \( C^1 \)-boundary and outer unit normal \( \nu \), \( f \in L^2(\Omega) \). For functions \( a \in L^\infty(\Omega, M_d(\mathbb{R})) \) and \( q \in L^\infty(\Omega) \) we define the operator \( L: W^{1,2}(\Omega) \to W^{1,2}(\Omega)' \) by

\[ (Lv)(u) = \int_{\Omega} [(a(x) \nabla v(x), \nabla u(x)) + q(x)v(x)u(x)] \, dx \text{ for } u,v \in W^{1,2}(\Omega). \]

Additionally \( a(\cdot) \) is symmetric and elliptic on \( \Omega \) with some constant \( \mu > 0 \). Show that:

1. The operator \( L \) is an isomorphism, if \( q(x) \geq \lambda > 0 \) for all \( x \in \Omega \) and some \( \lambda > 0 \).
2. The operator \( L \) is a Fredholm-Operator with index zero.
   (Hint: Use without proof that the embedding \( W^{1,2}(\Omega) \subseteq L^2(\Omega) \) is compact, if \( \Omega \subseteq \mathbb{R}^d \) is a bounded domain with \( C^1 \)-boundary)
3. The condition \( \int_{\Omega} f(x) \, dx = 0 \) is necessary and sufficient for the existence of a weak solution to the Neumann-Problem

\( \begin{cases} -\text{div}(a \nabla v) = f & \text{in } \Omega \\ \sum_{i=1}^{d} \nu_i \sum_{j=1}^{d} a_{ij} \partial_j v = 0 & \text{on } \partial\Omega \end{cases} \)

Exercise 3 (Solvability criteria)

Consider the operator \( L: W^{1,2}_0(\Omega) \to W^{1,2}(\Omega)' \), \( L = L_0 + K \) defined in the lecture with measurable coefficients \( a, b, c, q \) and let the operator \( L_0 \) be elliptic with constant \( \mu > 0 \). Show for every \( \varphi \in W^{1,2}_0(\Omega) \) the equivalence of the following statements (Theorem 10.20 in the lecture):

1. \( L\varphi = \varphi \) has a solution \( v \in W^{1,2}_0(\Omega) \).
2. \( \varphi(u) = 0 \) for all \( u \in \ker(\text{L}^*) \).

Here \( \text{L}^* := \text{L}' \circ J \), where \( J: W^{1,2}(\Omega) \to W^{1,2}(\Omega)'' \) is the canonical embedding. What is the meaning of (2), if the right-hand side is some \( L^2 \)-function \( f \)?

Exercise 4 ((C) The spectrum of a multiplication operator)

Let \( \emptyset \neq \Omega \subseteq \mathbb{R}^d \), \( X := C^0_b(\Omega) := \{ f: \Omega \to \mathbb{K} \mid f \text{ is continuous and bounded on } \Omega \} \), \( K \in \{ \mathbb{R}, \mathbb{C} \} \), \( m \in X \) and \( T_m \) be the multiplication operator on \( X \), i.e.

\( T_m: X \to X, \ f \mapsto m \cdot f. \)

What is the spectrum \( \sigma(T_m) \)? Determine the type of the spectral values and the resolvent \( R(\lambda, T_m) \) for all \( \lambda \in \rho(T_m) \).