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Functional analysis

14. Exercise Sheet

Exercise 1 ((C) Counterexample)

Let $I = [0,1] \subseteq \mathbb{R}$ be the closed unit interval and $K\colon L^2(I) \to L^2(I)$ be the operator defined by

$$(Kf)(x) = \int_0^x f(t)\mathrm{d}t,\ x \in I.$$

Show that the operator K is compact, but not self-adjoint and has no eigenvalues.

Exercise 2 (Inductive way to eigenvalues and eigenvectors)

Let $(X, \langle\cdot,\cdot\rangle_X)$ be a real Hilbert space, $K \in L(X,X)$ be a compact self-adjoint and positive semi-definite (i.e. $\langle Kx, x\rangle_X \geq 0$ for all $x \in X$) operator. Show that we can determine inductivly the eigenvalues λ_k and the eigenvectors v_k, $k \in \mathbb{N}$, with

$$\begin{aligned}\lambda_k &= \sup\left\{\langle Kx, x\rangle_X :\ \|x\|_X = 1,\ x \perp V_{k-1}\right\},\\ v_k &= \arg\sup\left\{\langle Kx, x\rangle_X :\ \|x\|_X = 1,\ x \perp V_{k-1}\right\},\\ V_k &= \operatorname{lin}\left\{v_1, \dots, v_k\right\} \text{ and } V_0 := \{0\}.\end{aligned}$$

Do we get all eigenvectors?

Exercise 3 (Again Hilbert-Schmidt-Integraloperators)

Let $X := L^2(I)$ with $I = [0,1] \subseteq \mathbb{R}$ and $K\colon L^2(I) \to L^2(I)$ be the Hilbert-Schmidt-Integraloperator

$$(Kf)(x) = \int_I k(x,y)f(y)\mathrm{d}y,\ x \in I, f \in L^2(I),$$

where

$$k(x,y) := \begin{cases}(1-x)y \text{ if } 0 \leq y \leq x\\ (1-y)x \text{ if } x < y \leq 1\end{cases},\ x, y \in I.$$

Show that the eigenvalues of K are $\left(\frac{1}{j\pi}\right)^2$, $j \in \mathbb{N}$, with eigenfunctions $v_j = \sqrt{2}\sin(j\pi\cdot)$.

The eigenfunctions are a (Hilbert-)basis of $L^2(I)$. Discuss with the ansatz $f = \sum_{j=1}^{\infty} c_j v_j$ the solvability of the equation

$$(K - \lambda \mathrm{Id}_X)\, v = f.$$