Karlsruhee Institut für Technologie
Institut für Analysis
Dr. A. Müller-Rettkowski
M. Sc. S. Avramska

SS 2012
24.04.2012

Funktionentheorie
Exercise Sheet 1
Solutions

Exercise 1

(a) (1) $(2+i)^3 = 2 + 11i$;
polar coordinates: $2 + 11i = re^{i\theta}$ with $r = \sqrt{2^2 + 11^2} = 5\sqrt{5}$, $\theta = \arctan \frac{11}{2}$;

(2) $\frac{3+4i}{1+(1-i)^2} = -1 + 2i$;
polar coordinates: $-1 + 2i = re^{i\theta}$ with $r = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$, $\theta = \pi - \arctan \frac{2}{1}$;

(b) (1) $z^3 - 3z^2 + 6z - 4 = (z-1)\big((z-1)^2 + 3\big) \Rightarrow z_1 = 1, z_{2,3} = 1 \pm i\sqrt{3}$;

(2) Let $z = x + iy$. Then

$$\begin{aligned} z^2 - 2\bar{z} + 1 &= (x+iy)^2 - 2(x - iy) + 1 \\ &= x^2 - y^2 - 2x + 1 + i(2xy + 2y). \end{aligned}$$

Therefore,

$$z^2 - 2\bar{z} + 1 = 0 \Leftrightarrow x^2 - y^2 - 2x + 1 = 0 \text{ and } 2xy + 2y = 0.$$

If $y = 0$, from the first equation we get $x_{1,2} = 1$, and so $z_{1,2} = 1$.
If $y \neq 0$, from the second equation we get $x = -1$ and then from the first equation we conculde that $y = \pm i2$. Thus $z_{1,2} = -1 \pm 2i$.

(3) We have

$$\frac{2 + 10i}{2 - 3i} = -2 + 2i = 2\sqrt{2}e^{i\frac{3}{4}\pi}.$$

We write $z = re^{i\theta}$. The given equation is then equivalent to

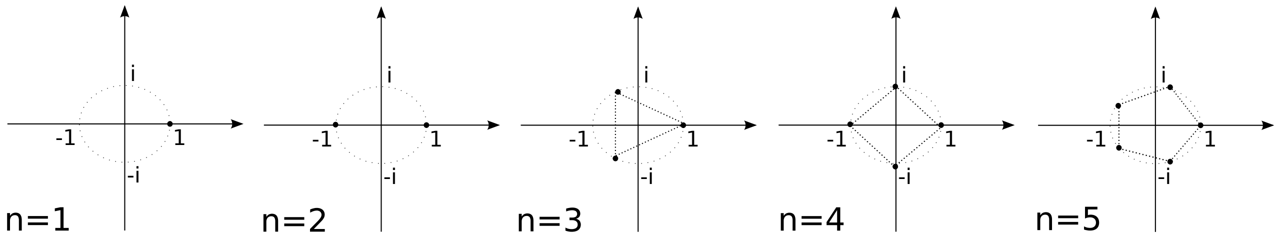
$$r^3 e^{3i\theta} = 2\sqrt{2}e^{i\frac{3}{4}\pi}.$$

Therefore, $r = \sqrt{2}, \theta = \frac{\pi}{4} + \frac{2}{3}n\pi, n \in \mathbb{Z}$. Since by definition $\theta \in (-\pi, \pi]$, it follows that $n = -1, 0, 1$ and thus

$$\begin{aligned} z_1 &= \sqrt{2}e^{i\left(\frac{\pi}{4} - \frac{2}{3}\pi\right)} = \sqrt{2}e^{-i\frac{5}{12}\pi}, \\ z_2 &= \sqrt{2}e^{i\frac{\pi}{4}}, \\ z_1 &= \sqrt{2}e^{i\left(\frac{\pi}{4} + \frac{2}{3}\pi\right)} = \sqrt{2}e^{i\frac{11}{12}\pi}. \end{aligned}$$

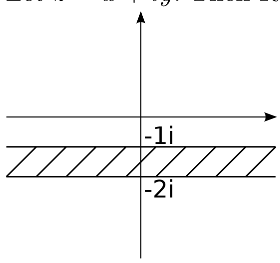
Exercise 2

(a) The n-th roots of unity for $n = 1, \ldots, 5$ look like as follows

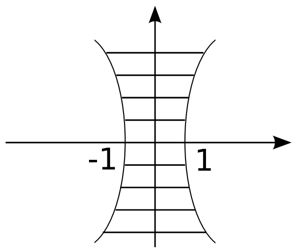


All n-th roots of unity for $n \in \mathbb{N}$ lie at the unit circle and are vertices of a regular n-gon inscribed in the unit circle. The point $(1, 0)$ is always one of the vertices.

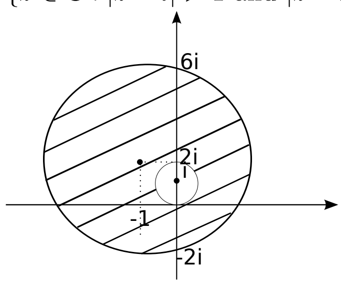
(b) (1) Let $z = x + iy$. Then $\text{Re}(iz) = \text{Re}(ix - y) = y$. Therefore $1 < \text{Re}(iz) < 2 \Leftrightarrow 1 < y < 2$.



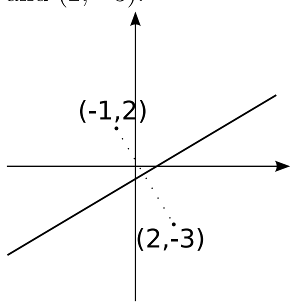
(2) Let $z = x + iy$. Then $\text{Re}(iz) = \text{Re}(x^2 - y^2 + 2ixy) = x^2 - y^2$. Therefore $\text{Re}(z^2) < 1 \Leftrightarrow x^2 - y^2 < 1$.



(3) We know that in the complex plane $|z - a| = r$ is a circle with center a and radius r. Therefore $\{z \in \mathbb{C} : |z - i| > 1 \text{ and } |z - 2i + 1| \leq 4\}$ is



(4) $\{z \in \mathbb{C} : |z - 2i + 1| = |z + 3i - 2|\}$ means that z has to be at equal distance from the points $(1, -2)$ and $(2, -3)$.



Exercise 3

$$\begin{aligned}
\prod_{k=1}^{n-1}\sin\frac{k\pi}{n} &= \prod_{k=1}^{n-1}\frac{e^{i\frac{k\pi}{n}}-e^{-i\frac{k\pi}{n}}}{2i}\\
&= \frac{1}{(2i)^{n-1}}\prod_{k=1}^{n-1}e^{-i\frac{k\pi}{n}}\prod_{k=1}^{n-1}\Big(e^{2i\frac{k\pi}{n}}-1\Big)\\
&= \frac{1}{(2i)^{n-1}}e^{-i\frac{\pi}{n}\frac{n(n-1)}{2}}\prod_{k=1}^{n-1}\Big(e^{2i\frac{k\pi}{n}}-1\Big)\\
&= \frac{1}{(2i)^{n-1}}(-i)^{n-1}\prod_{k=1}^{n-1}\Big(e^{2i\frac{k\pi}{n}}-1\Big)\\
&= (-2)^{1-n}\prod_{k=1}^{n-1}\Big(e^{2i\frac{k\pi}{n}}-1\Big).\\
&= 2^{1-n}\prod_{k=1}^{n-1}\Big(1-\zeta^k\Big)\quad\text{with } \zeta=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}. \qquad (1)
\end{aligned}$$

From the hint we know that

$$\begin{aligned}
z^n-1=\prod_{k=1}^{n}(z-\zeta^k)=(z-\zeta^n)\prod_{k=1}^{n-1}(z-\zeta^k)&=(z-1)\prod_{k=1}^{n-1}(z-\zeta^k)\\
\frac{z^n-1}{z-1}&=\prod_{k=1}^{n-1}(z-\zeta^k).
\end{aligned}$$

Letting $z\to 1$, we get

$$\prod_{k=1}^{n-1}(1-\zeta^k)=n \qquad (2)$$

Combining (1) and (2), we get the desired result.

Exercise 4 First, let us assume that the sequence is convergent and it converges to some $z\in\mathbb{C}$. Letting $n\to\infty$ on both sides of

$$z_{n+1}=\frac{1}{2}\Big(z_n+\frac{1}{z_n}\Big)$$

we can conclude that the possible limits are $z=\pm1$.
Next, we note that if z_0 lies in the right-half plane ($x_0>0$), then inductivelly all z_n also lie in the right-half plane

$$\frac{1}{2}\Big(z_n+\frac{1}{z_n}\Big)=\frac{1}{2}\Big(\frac{x_n(x_n^2+y_n^2+1)}{x_n^2+y_n^2}+i\frac{y_n(x_n^2+y_n^2-1)}{x_n^2+y_n^2}\Big)$$

Analogously, if z_0 lies in the left-half plane ($x_0<0$), then inductivelly all z_n also lie in the left-half plane. A special case is when z_0 lies on the imaginary axis. Then all z_n are purely imaginary (if $z_0=ai, a\in\mathbb{R}$ then $z_1=\frac{1}{2}(ai+\frac{1}{ai})=\frac{a^2-1}{2a}i$; inductivelly all $z_n, n\in\mathbb{N}$ are imaginary) or undefined (if $z_n=0$ then z_{n+1} is undefined). Hence the sequence cannot converge to ±1, so it is divergent.
Without loss of generality, we can assume that z_0 lies in the right half-plane. Now we consider the auxiliary sequence

$$w_{n+1}=\frac{z_{n+1}-1}{z_{n+1}+1}.$$

We note that it holds $w_{n+1}=(w_n)^2$. Furthermore, we have $|w_0|<1$. Thus, $\{w_n\}_{n\in\mathbb{N}}$ is convergent and it converges to 0. Since $w_{n+1}=1-\frac{2}{z_{n+1}+1}$, it follows that $z_n\to 1$.