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Funktionentheorie
Exercise Sheet 10
Solutions

Exercise 1

(a) First we note that since $|f(z)| \leq 1$ for $|z| < 1$, the points

$$w = g(z) = \frac{f(z) - a}{1 - \bar{a} f(z)}, \quad |a| < 1,$$

will satisfy $|w| \leq 1$. Hence $g(z)$ is also a regular and bounded analytic function for $|z| < 1$. Since $|f(0)| < 1$ (otherwise, if we assume that $f(0) = 1$, by the maximum principle, it follows that $f = const =: c$ with $|c| \leq 1$ and then the inequalities are trivial), we may take $a = f(0)$. We thus obtain the bounded function

$$g(z) = \frac{f(z) - f(0)}{1 - \overline{f(0)} f(z)}, \tag{1}$$

which, obviously, vanishes for $z = 0$. Thus we may apply the Schwarz lemma and we get

$$|g(z)| \leq |z|.$$

Solving (1) for $f(z)$, we obtain

$$f(z) = \frac{f(0) + g(z)}{1 + \overline{f(0)} g(z)}.$$

Thus, using the inequalities from the hint

$$\frac{|b| - |d|}{1 - |b||d|} \leq \frac{|b + d|}{1 + |\bar{b} d|} \leq \frac{|b| + |d|}{1 + |b||d|}, \quad |b| < 1, |d| < 1 \tag{2}$$

for $b = f(0)$ and $d = g(z)$, we get the desired result.

(b) From the Schwarz lemma we know that if $f : D \to D$ is holomorph with $f(a) = b$, then it follows that

$$|f'(a)| \leq \frac{1 - |b|^2}{1 - |a|^2}.$$

For $a = 1/2$ and $b = 3/4$ we have

$$\left| f'\left(\frac{1}{2}\right) \right| \leq \frac{7}{12},$$

which is a contradiction, since we have $f'(1/2) = 2/3$.

Exercise 2 We start by considering $|f|$ on the circle $|z| = R$. We introduce the auxiliary function

$$g(z) = (z^2 - R^2) f(z).$$

For every z with $|z| = R$ and $Rez \geq 0$ we have

$$|(z - R) f(z)| \leq \frac{|z - R|}{|Imz|} = sec\theta$$

for some $0 \leq \theta \leq \pi/4$. Thus,

$$|(z - R) f(z)| \leq \sqrt{2}.$$

Similarly, if $|z| = R$ and $Rez \leq 0$, then

$$|(z+R)f(z)| \leq \sqrt{2}.$$

Hence

$$|g(z)| = |z^2 - R^2||f(z)| \leq 3R$$

for all z with $|z| = R$. By the Maximum principle, the same upper bound holds throughout $|z| < R$. Hence

$$|f(z)| \leq \frac{3R}{|z^2 - R^2|}$$

as long as $|z| \leq R$. Letting $R \to \infty$, we see that $f(z) = 0$.

Exercise 3 First we note that $\tilde{f}$ is holomorph in G_-. For $z \in G_-$ we have

$$\frac{\tilde{f}(z+h) - \tilde{f}(z)}{h} = \frac{\overline{f(\bar{z}+\bar{h}) - f(\bar{z})}}{h} = \overline{\Big[\frac{f(\bar{z}+\bar{h}) - f(\bar{z})}{\bar{h}}\Big]} \underset{h\to 0}{\longrightarrow} \overline{f'(\bar{z})}.$$

Next we show that $\tilde{f}$ is countinuous in G. This is clear for all points from $G_+ \cup G_-$. So let $z \in G_0$ (i.e. $z \in \mathbb{R}$ and $f(z) \in \mathbb{R}$). Moreover, let $\{z_n\}_{n\in\mathbb{N}} \subset G_-$ with $z_n \underset{n\to\infty}{\longrightarrow} z$. Then it also holds $\{\bar{z}_n\}_{n\in\mathbb{N}} \subset G_+$ with $\bar{z}_n \underset{n\to\infty}{\longrightarrow} \bar{z} = z$. From the continuity of f in $G_+ \cup G_0$ it follows that $f(\bar{z}_n) \underset{n\to\infty}{\longrightarrow} f(\bar{z}) = f(z)$. Hence

$$\tilde{f}(z_n) = \overline{f(\bar{z}_n)} \underset{n\to\infty}{\longrightarrow} \overline{f(\bar{z})} = f(\bar{z}) = f(z) = \tilde{f}(z).$$

This together with the continuity of f in $G_+ \cup G_0$ allows us to conclude that $\tilde{f}$ is continuous in G.

In order to show that $\tilde{f}$ is holomorph, we will use the Morera's theorem. So let $\Delta = [a, b, c, a]$ be a triangle with $\Delta \subset G$. We have to show that $\int\limits_\Delta \tilde{f}(z)dz = 0$. We note that if $\Delta \subset G_+$ or $\Delta \subset G_-$ then, from the Goursat's lemma, it follows that $\int\limits_\Delta \tilde{f}(z)dz = 0$. Thus, we have to consider $\Delta \subset G_+ \cup G_0 \cup G_-$. Let us first assume that $\Delta \subset G_+ \cup G_0$ with $[a, b] \subset G_0$. By hypothesis f is continuous on $G_+ \cup G_0$ and so f is uniformly continuous in Δ with its interior, i.e. if $\varepsilon > 0$ there exists $\delta > 0$ such that when $|z - z'| < \delta$ then $|f(z) - f(z')| < \varepsilon$. Now choose $\alpha \in [a, c]$ and $\beta \in [b, c]$ so that $|a - \alpha| < \delta$ and $|b - \beta| < \delta$. Let $T = [\alpha, \beta, c, \alpha]$ and $Q = [a, b, \beta, \alpha, a]$. Then

$$\int\limits_\Delta f = \int\limits_T f + \int\limits_Q f.$$

But since T with its interior is contained in G_+ and f is analytic there, it follows that

$$\int\limits_\Delta f = \int\limits_Q f.$$

We note that

$$\Big|\int\limits_{[\alpha,a]} f\Big| \leq M|\alpha - a| \leq M\delta, \tag{3}$$

where $M = \max\{|f(z)| : z \in \Delta\}$. Analogously

$$\Big|\int\limits_{[b,\beta]} f\Big| \leq M\delta. \tag{4}$$

Furthermore, if $t \in [0, 1]$ then

$$|[t\beta + (1-t)\alpha] - [tb + (1-t)a]| < \delta$$

and so

$$|f(t\beta + (1-t)\alpha) - f(tb + (1-t)a)| < \varepsilon.$$

Thus

$$\Big|\int\limits_{[a,b]} f + \int\limits_{[\alpha,\beta]} f\Big| = \Big|(b-a)\int\limits_0^1 f(tb+(1-t)a)dt - (\beta-\alpha)\int\limits_0^1 f(t\beta+(1-t)\alpha)dt\Big|$$

$$\le |b-a|\Big|\int\limits_0^1 \big[f(tb+(1-t)a) - f(t\beta+(1-t)\alpha)\big]dt + |(b-a)-(\beta-\alpha)|\Big|\int\limits_0^1 f(t\beta+(1-t)\alpha)dt\Big|$$

$$\le |b-a|\varepsilon + M|(b-\beta)+(\alpha-a)|$$

$$\le l\varepsilon + 2M\delta,$$

where l is the perimeter of Δ. Combining this estimate with (3) and (4) we get

$$\Big|\int\limits_{\Delta} f\Big| \le l\varepsilon + 4M\delta.$$

Since it is possible to choose $\delta < \varepsilon$ and ε is arbitrary, it follows that $\int\limits_{\Delta} f = 0$.

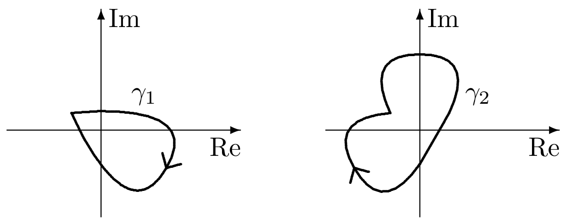
Analogously, it can be proved that $\int\limits_{\Delta} f = 0$ for every $\Delta \subset G_+ \cup G_0$ or $\Delta \subset G_- \cup G_0$.

Next we show that any arbitrary triangle Δ can be transformed into $\Delta \subset G_+ \cup G_0$ with $[a,b] \subset G_0$. Suppose $a, b \in G_-, c \in G_+$. Let s_a and s_b be the points where the lines $[a,c]$ and $[b,c]$, resp., intersect the real axis, and let $\Delta_a := \Delta(a,b,s_a) \subset G_- \cup G_0$, $\Delta_b := \Delta(b,s_b,s_a) \subset G_- \cup G_0$ and $\Delta_+ := \Delta(s_b,c,s_a) \subset G_+ \cup G_0$. Then it holds

$$\int\limits_{\Delta} \tilde{f}(z)\,dz = \int\limits_{\Delta_a} \tilde{f}(z)\,dz + \int\limits_{\Delta_b} \tilde{f}(z)\,dz + \int\limits_{\Delta_+} \tilde{f}(z)\,dz = 0+0+0 = 0.$$

Exercise 4

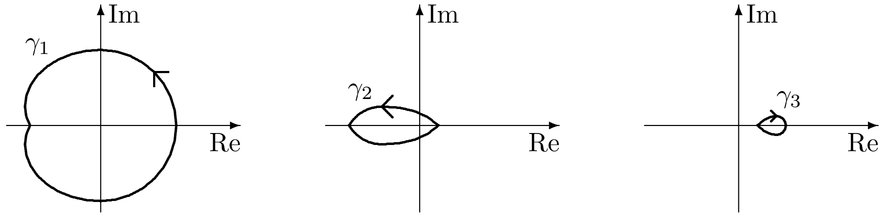
(i) The given curve can be considered as union of the following two curves



Since these curves have a common point, it follows

$$\int\limits_{\gamma} \frac{1}{z}\,dz = -\int\limits_{-\gamma_1} \frac{1}{z}\,dz - \int\limits_{-\gamma_2} \frac{1}{z}\,dz = -4\pi i.$$

(ii) For γ_k $(k=1,2,3)$ as follows



it holds

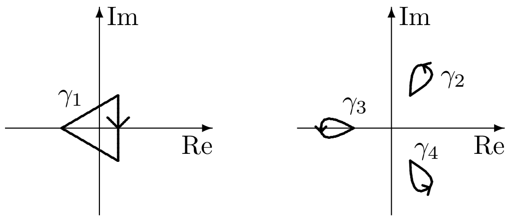
$$\int\limits_{\gamma} \frac{1}{z}\,dz = \sum_{k=1}^{3} \int\limits_{\gamma_k} \frac{1}{z}\,dz = 2\pi i + 2\pi i + 0 = 4\pi i,$$

since the function f is holomorph in $\{z \in \mathbb{C} : Rez > 0\}$ and $\gamma_3 \subset \{z \in \mathbb{C} : Rez > 0\}$.

(iii) In this case we have

$$\int_{\gamma} \frac{1}{z}\,dz = \sum_{k=1}^{4} \int_{\gamma_k} \frac{1}{z}\,dz,$$

for



Analogously to (ii), we can conclude that

$$\int_{\gamma} \frac{1}{z}\,dz = -2\pi i + 0 + 0 + 0 = -2\pi i$$

(iv) Since $\gamma \subset \mathbb{C} \setminus \{z \in i\mathbb{R} : Imz \geq 0\}$ and the function $z \mapsto 1/z$ is holomorph there, from the Cauchy integral theorem, it follows that

$$\int_{\gamma} \frac{1}{z}\,dz = 0$$