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Funktionentheorie
Exercise Sheet 11
Solutions

Exercise 1 Since f is the uniform limit of the continuous functions $f_1, f_2, \dots$ in every compact subset of G, it follows that f is continuous in every compact subset of G, and so in the whole of G.
Let $z_0 \in G$ and let $K \subset G$ be a neighbourhood of z_0. For every closed path $\gamma \subset K$, the Cauchy integral theorem gives us

$$\int_\gamma f_n(z)dz = 0 \quad \text{for all } n \in \mathbb{N}.$$

Since $|\gamma|$ is a compact set and $f_n \xrightarrow{n\to\infty} f$ uniformly in every compact subset of G, we can interchange the integral and the limit signs and so

$$\int_\gamma f(z)dz = \int_\gamma \lim_{n\to\infty} f(z)dz = \lim_{n\to\infty} \int_\gamma f_n(z)dz = 0.$$

Due to the fact that $\int_\gamma f$ is locally independent of the path of integration, we may choose γ to be a triangle and so, using the theorem of Morrera, we can conclude that f is holomorph. Therefore all the derivatives $f_n^{(k)} k \in \mathbb{N}$ exist.
Now let $z_0 \in G$ be arbitrary. We show that there exists an open ball $B := B(z_0, r) \subset G$ on which $f_n^{(k)} \xrightarrow{n\to\infty} f^{(k)}$ uniformly. To this end, choose $r > 0$ so small that the closed circle $B_2 := \{z \in \mathbb{C} : |z - z_0| \le 2r\} \subset G$. Hence, for every $z \in B_2$ (and thus for every $z \in B$), we have

$$f_n^{(k)}(z) = \frac{k!}{2\pi i} \int_{B_2} \frac{f_n(\xi)}{(\xi - z)^{k+1}} d\xi, \quad f^{(k)}(z) = \frac{k!}{2\pi i} \int_{B_2} \frac{f(\xi)}{(\xi - z)^{k+1}} d\xi.$$

Therefore

$$\begin{aligned}
\left|f_n^{(k)}(z) - f^{(k)}(z)\right| &= \left|\frac{k!}{2\pi i} \int_{B_2} \frac{f_n(\xi) - f(\xi)}{(\xi - z)^{k+1}} d\xi\right| \\
&\le \frac{k!}{2\pi} 4\pi r \max_{B_2} \frac{|f_n(\xi) - f(\xi)|}{|\xi - z|^{k+1}} \\
&\le 2rk! \max_{B_2} \frac{|f_n(\xi) - f(\xi)|}{r^{k+1}} \quad (z \in B \text{ and } |\xi - z_0| = 2r \Rightarrow |\xi - z| \ge r) \\
&= \frac{2k!}{r^k} \max_{B_2} |f_n(\xi) - f(\xi)|.
\end{aligned}$$

Thus, from the uniform convergence of $\{f_n\}_{n\in\mathbb{N}}$ to f on the compact set B_2 follows the uniform convergence $\{f_n^{(k)}\}_{n\in\mathbb{N}}$ to $f^{(k)}$ on B.

We know now: for every point $z \in G$ there exists a neighbourhood $B(z)$ on which $\{f_n^{(k)}\}_{n\in\mathbb{N}}$ converges uniformly to $f^{(k)}$. So if we take an arbitrary compact set $M \subset G$, $\{B(z) : z \in M\}$ is an open cover of M. Since M is compact it can be covered by finitely many $B(z)$s. Therefore, $\{f_n^{(k)}\}_{n\in\mathbb{N}}$ converges uniformly to $f^{(k)}$ on the union of these $B(z)$s and thus also on M.

Exercise 2

(a) Assume that there exists a point $a \in \mathbb{C}$ such that there is no sequence $\{z_n\}_{n\in\mathbb{N}}$ with $f(z_n) \xrightarrow{n\to\infty} a$. Then there exists $\varepsilon > 0$ such that

$$|f(z) - a| \ge \varepsilon \quad \text{for all } z \in \mathbb{C}.$$

Then $h := (f - a)^{-1}$ is holomorph in $\mathbb{C}$ and $|h| \le \varepsilon^{-1}$. Thus, by the theorem of Liouville, h is constant and so is f, which is a contradiction.

(b) If we assume that $f \neq const$ then, by (a), $f(\mathbb{C})$ must contain points from the upper half-plane as well as points from the lower half-plane. By the Open mapping theorem (Theorem 4 in Chapter 10.4) it follows that $f(\mathbb{C})$ is a domain, i.e. it is connected. But then it must contain points from $\mathbb{R}$, which is in contradiction with the hypothesis of the exercise.

(c) In the power series representation of f around z_0 the coefficients are

$$a_n = \frac{f^{(n)}}{n!}.$$

By hypothesis, for every z_0 there exists $n \in \mathbb{N}_0$ s.t. $f^{(n)}(z_0) = 0$. Let $D := \{z \in \mathbb{C} : |z| \leq 71\}$ and $D_n := \{z \in D : f^{(n)}(z) = 0\}$. Then

$$D = \bigcup_{n=0}^{\infty} D_n.$$

For fixed $n_0 \in \mathbb{N}_0$ D_{n_0} contains infinitely many points, because otherwise D as a countable union of finite set will be countable. Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in D with pairwise different points. Since it is bounded, it contains a convergent subsequence $\{z_{k_l}\}_{l\in\mathbb{N}}$ with $z_{k_l} \xrightarrow{l\to\infty} z \in D_{n_0} \subset \mathbb{C}$. Since $f^{(n_0)}(z_{k_l}) = 0$ for all $l \in \mathbb{N}$, by the Identity theorem, we can conclude that $f^{(n_0)}(z) = 0$ for all $z \in \mathbb{C}$ and thus f is a polynomial.

Exercise 3

$$\begin{aligned}
\int_\gamma \Big(\frac{z}{z-1}\Big)^n dz &= \int_0^{2\pi} \Big(\frac{1+e^{it}}{e^{it}}\Big)^n ie^{it}dt \\
&= \int_0^{2\pi} i\sum_{k=0}^{n}\binom{n}{k} e^{it(1-k)}dt \\
&= \sum_{k=0}^{n} i\binom{n}{k}\int_0^{2\pi} e^{it(1-k)}dt \\
&= i\binom{n}{1}\int_0^{2\pi} 1dt = in2\pi
\end{aligned}$$

Exercise 4 We first claim that each F_m is continuous. In fact, by induction we can show that the following factorization holds

$$\begin{aligned}
\frac{1}{(w-a)^m} - \frac{1}{(w-b)^m} &= \Big[\frac{1}{w-a} - \frac{1}{w-a}\Big]\sum_{k=1}^{m}\frac{1}{(w-a)^{m-k}}\frac{1}{(w-b)^{k-1}} \\
&= (a-b)\Big[\frac{1}{(w-a)^m(w-b)} + \frac{1}{(w-a)^{m-1}(w-b)^2} + \cdots + \frac{1}{(w-a)(w-b)^m}\Big]
\end{aligned}$$

Moreover, since $|\gamma|$ is compact, it follows that φ is bounded there. Let $\overline{\varphi} := \max_\gamma \varphi$. Fix a and let $r := dist(a, |\gamma|)$. Then, if $|a-b| < \delta < \frac{1}{2}r$, we have

$$\begin{aligned}
|F_m(a) - F_m(b)| &= \Big|\int_\gamma \Big[\frac{\varphi(w)}{(w-a)^m} - \frac{\varphi(w)}{(w-b)^m}\Big]dw\Big| \\
&\leq \overline{\varphi}|a-b|\int_\gamma |dw|\Big[\frac{1}{|w-a|^m|w-b|} + \frac{1}{|w-a|^{m-1}|w-b|^2} + \cdots + \frac{1}{|w-a||w-b|^m}\Big]
\end{aligned}$$

We note that for $|a-b| < \frac{1}{2}r$ and w on $|\gamma|$ we have that $|w-a| > \frac{1}{2}r$ and $|w-b| > \frac{1}{2}r$. Hence

$$|F_m(a) - F_m(b)| \leq \overline{\varphi}\delta m\frac{1}{(\frac{1}{2}r)^{m+1}}l(\gamma)$$

So if $\varepsilon > 0$ is given, by choosing $\delta < \min(\frac{1}{2}r, \frac{(\frac{1}{2}r)^{m+1}}{m\overline{\varphi}l(\gamma)})$, we see that F_m must be continuous.

Now fix b in $G := \mathbb{C} \setminus |\gamma|$ and let $a \in G, a \neq b$. We have

$$\frac{F_m(a) - F_m(b)}{a-b} = \int_\gamma (w-b)^{-1}\frac{\varphi(w)}{(w-a)^m}dw + \cdots + \int_\gamma (w-b)^{-m}\frac{\varphi(w)}{(w-a)}dw \tag{1}$$

Since $b \notin |\gamma|$, it follows that $(w-b)^{-k}\varphi(w)$ is continuous on $|\gamma|$ for each k. Therefore each integral on the right-hand side of (1) is continuous function of $a, a \in G$. Hence, letting $a \to b$, we can conclude that

$$\begin{aligned} F_m'(b) &= \int_\gamma \frac{\varphi(w)}{(w-b)^{m+1}} dw + \cdots + \int_\gamma \frac{\varphi(w)}{(w-b)^{m+1}} dw \\ &= mF_{m+1}(b). \end{aligned}$$