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SS 2012
10.07.2012

Funktionentheorie
Exercise Sheet 12
Solutions

Exercise 1 First we note: If we want to expand

$$\frac{1}{z-a}$$

around the point $z_0 \neq a$ in its Lorents series representation, we have the following two possibilities:
For $|z-z_0| < |a-z_0|$ we have

$$\begin{aligned}\frac{1}{z-a} &= \frac{1}{(z-z_0)+(z_0-a)} = \frac{1}{z_0-a}\cdot\frac{1}{\frac{z-z_0}{z_0-a}+1}\\ &= \frac{1}{z_0-a}\cdot\frac{1}{1-\frac{z-z_0}{a-z_0}} = \frac{1}{z_0-a}\sum_{k=0}^{\infty}\Big(\frac{z-z_0}{a-z_0}\Big)^k\\ &= -\sum_{k=0}^{\infty}\frac{(z-z_0)^k}{(a-z_0)^{k+1}}. \qquad (1)\end{aligned}$$

For $|z-z_0| > |a-z_0|$ we have

$$\begin{aligned}\frac{1}{z-a} &= \frac{1}{(z-z_0)+(z_0-a)} = \frac{1}{z-z_0}\cdot\frac{1}{1+\frac{z_0-a}{z-z_0}}\\ &= \frac{1}{z-z_0}\cdot\frac{1}{1-\frac{a-z_0}{z-z_0}} = \frac{1}{z-z_0}\sum_{k=0}^{\infty}\Big(\frac{a-z_0}{z-z_0}\Big)^k\\ &= \sum_{k=0}^{\infty}\frac{(a-z_0)^k}{(z-z_0)^{k+1}}. \qquad (2)\end{aligned}$$

(a) For $1 < |z| < 3$ we have

$$\begin{aligned}f(z) &= \frac{1}{1-z^2} + \frac{1}{3-z} = \frac{1}{z^2}\cdot\frac{1}{z^{-2}-1} - \frac{1}{z-3}\\ &= -\frac{1}{z^2}\cdot\frac{1}{1-z^{-2}} - \frac{1}{z-3} = -\frac{1}{z^2}\sum_{k=0}^{\infty}(z^{-2})^k - \frac{1}{z-3}\\ &= -\sum_{k=0}^{\infty}\frac{1}{z^{2k+2}} - \frac{1}{z-3} \overset{(1)}{=} -\sum_{k=0}^{\infty}\frac{1}{z^{2(k+1)}} + \sum_{k=0}^{\infty}\frac{z^k}{3^{k+1}}.\end{aligned}$$

(b) We can write f in the following form

$$f(z) = \frac{1}{2}\Big(\frac{1}{z+1} - \frac{1}{z-1}\Big) - \frac{1}{z-3}.$$

For $|z-2| < 3$, by (1), it holds

$$\frac{1}{z+1} = \frac{1}{z-(-1)} = -\sum_{k=0}^{\infty}\frac{(z-2)^k}{(-1-2)^{k+1}} = -\sum_{k=0}^{\infty}\frac{(z-2)^k}{(-3)^{k+1}},$$

and for $|z-2| > 1$, by (2), it holds

$$\begin{aligned}\frac{1}{z-1} &= \sum_{k=0}^{\infty}\frac{(1-2)^k}{(z-2)^{k+1}} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(z-2)^{k+1}}\\ \frac{1}{z-3} &= \sum_{k=0}^{\infty}\frac{(3-2)^k}{(z-2)^{k+1}} = \sum_{k=0}^{\infty}\frac{(1)^k}{(z-2)^{k+1}}.\end{aligned}$$

Altogether

$$\begin{aligned} f(z) &= -\frac{1}{2}\sum_{k=0}^{\infty}\frac{(z-2)^k}{(-3)^{k+1}} - \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(z-2)^{k+1}} - \sum_{k=0}^{\infty}\frac{(1)^k}{(z-2)^{k+1}} \\ &= \sum_{k=0}^{\infty}\frac{\frac{1}{2}(-1)^{k+1}-1}{(z-2)^{k+1}} - \sum_{k=0}^{\infty}\frac{(z-2)^k}{2(-3)^{k+1}}. \end{aligned}$$

Exercise 2 For $|z| < 1$ we have

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z}\frac{1}{1-z} = -\frac{1}{z}\sum_{k=0}^{\infty} z^k = -\sum_{k=-1}^{\infty} z^k.$$

Therefore

$$f'(z) = -\sum_{k=-1}^{\infty} kz^{k-1} = -\sum_{k=-2}^{\infty}(k+1)z^k.$$

Analogously, for $|z| > 1$ we have

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{z^2}\frac{1}{1-\frac{1}{z}} = \frac{1}{z^2}\sum_{k=0}^{\infty}\Big(\frac{1}{z}\Big)^k = \sum_{k=2}^{\infty} z^{-k}.$$

Therefore

$$f'(z) = \sum_{k=2}^{\infty}(-k)z^{-k-1} = \sum_{k=3}^{\infty}(-k+1)z^{-k}.$$

Exercise 3

(a) Let

$$\sum_{n=0}^{\infty} a_n z^n$$

be the power series expansion of the function $f(z) = \frac{1}{1-z-z^2}$ in $\{z \in \mathbb{C} : |z| > \frac{2}{\sqrt{5}-1}\}$ (there f is holomorph). Then we have

$$\begin{aligned} (1-z-z^2)\sum_{n=0}^{\infty} a_n z^n &= 1 \\ a_0 + (a_1 - a_0)z - \sum_{n=2}^{\infty}(a_n - a_{n-1} - a_{n-2})z^n &= 1 \end{aligned}$$

Therefore $a_1 = a_0 = 1$ and $a_n = a_{n-1} + a_{n-2}$, i.e. the coefficients a_n satisfy the same relations as f_n. Since we know that the power series expansion of an analytic funtion is unique, it follows that $a_n \equiv f_n$.

(b) We use the following fractional decomposition of f

$$\frac{1}{1-z-z^2} = \frac{\frac{1+\sqrt{5}}{2\sqrt{5}}}{1-\frac{1+\sqrt{5}}{2}z} - \frac{\frac{1-\sqrt{5}}{2\sqrt{5}}}{1-\frac{1-\sqrt{5}}{2}z}.$$

Since we are in $\{z \in \mathbb{C} : |z| < \frac{2}{1-\sqrt{5}}\}$, we have

$$\begin{aligned} \frac{1}{1-z-z^2} &= \frac{1+\sqrt{5}}{2\sqrt{5}}\sum_{n=0}^{\infty}\Big(\frac{1+\sqrt{5}}{2}z\Big)^n - \frac{1-\sqrt{5}}{2\sqrt{5}}\sum_{n=0}^{\infty}\Big(\frac{1-\sqrt{5}}{2}z\Big)^n \\ &= \Big[\frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\Big(\frac{1+\sqrt{5}}{2}\Big)^{n+1} - \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\Big(\frac{1+\sqrt{5}}{2}\Big)^{n+1}\Big]z^n \end{aligned}$$

Thus

$$f_n = \frac{1}{\sqrt{5}}\Big(\frac{1+\sqrt{5}}{2}\Big)^{n+1} - \frac{1}{\sqrt{5}}\Big(\frac{1+\sqrt{5}}{2}\Big)^{n+1}$$

Exercise 4 Let

$$f_a(z) := e^{\frac{a}{2}(z-\frac{1}{z})}.$$

Since f_a is holomorph in $0 < |z| < \infty$, it can be expanded in its Laurent series. Then, for every $r \in (0, \infty)$ it holds

$$J_n(a) = \frac{1}{2\pi i} \oint_{|z|=r} \frac{f_a(z)}{z^{n+1}} dz$$

We choose $r = 1$. Using the parametrization $z(t) = e^{it}, t \in [0, 2\pi]$, we get

$$J_n(a) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{f_a(z)}{z^{n+1}} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f_a(e^{it})}{e^{i(n+1)t}} i e^{it} dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{f_a(e^{it})}{e^{int}} dt.$$

Noticing that $e^{it} - \frac{1}{e^i t} = 2i \sin t$, we get

$$J_n(a) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{ia \sin t}}{e^{int}} dt = \frac{1}{2\pi} \Big(\int_0^{\pi} e^{i(a \sin t - nt)} dt + \int_{\pi}^{2\pi} e^{i(a \sin t - nt)} dt \Big).$$

In the second integral we make the substitution $s = 2\pi - t$. Then

$$\int_{\pi}^{2\pi} e^{i(a \sin t - nt)} dt = \int_{\pi}^{0} e^{i(a \sin(2\pi - s) - n(2\pi - s))} (-1) ds = \int_0^{\pi} e^{-ia \sin s + ins} ds.$$

Therefore

$$J_n(a) = \frac{1}{2\pi} \Big(\int_0^{\pi} e^{i(a \sin t - nt)} dt + \int_0^{\pi} e^{-i(a \sin t - nt)} dt \Big) = \frac{1}{\pi} \int_0^{\pi} \cos(a \sin t - nt) dt.$$