Karlsruher Institut für Technologie
Institut für Analysis
Dr. A. Müller-Rettkowski
M. Sc. S. Avramska

SS 2012
17.07.2012

Funktionentheorie
Exercise Sheet 13
Solutions

removable singularity = hebbare Singularität
essential singularity = wesentliche Singularität

Remark: If f has a pole of order m in a, then f can be written as

$$f(z) = \frac{g(z)}{(z-a)^m}$$

where g is a holomorphic function with $g(a) \neq 0$.

Proof: By definition, f has a pole of order m in a if

$$f(z) - \Big(\frac{a_{-m}}{(z-a)^m} + \cdots \frac{a_{-1}}{z-z_0}\Big)$$

has a removable singularity, i.e. if

$$f(z) - \Big(\frac{a_{-m}}{(z-a)^m} + \cdots \frac{a_{-1}}{z-z_0}\Big) = r(z),$$

with r holomorphic. Therefore

$$f(z) = \frac{1}{(z-a)^m}(a_{-m} + a_{-m+1}(z-a) + \cdots a_{-1}(z-a)^{m-1} + r(z)(z-a)^m) = \frac{g(z)}{(z-a)^m},$$

with g holomorphic.

Exercise 1

(a) The denominator of f has two zeroes: $z_1 = 4$ and $z_2 = -3$. In all other points the function f is holomorphic. Therefore, $z_{1,2}$ are isolated singularities. We have $f(z) = \frac{A(z)}{B(z)}$ for $A(z) = 1/(z+3)$ and $B(z) = (z-4)$. Since A is holomorphic in z_1 and $B(z_1) = 0$, it follows that z_1 is a pole of order 1. Analogously we can show that $z_2 = -1$ is also a pole of f.

(b) The denominator of f has zeroes in $z_{1,2} = \pm 2i$. We have $f(z) = \frac{A(z)}{B(z)}$ for $A(z) = 1/(z+2i)^2$ and $B(z) = (z-2i)^2$. Since A is holomorphic in $z_1 = 2i$ and z_1 is a zero of multiplicity 2 for $B(z)$, it follows that z_1 is a pole of order 2 for f. Analogously we can show that $z_2 = -2i$ is also a pole of order 2 for f. Obviously f has no other singularities.

(c) The function f is holomorphic in $\mathbb{C} \setminus \{0\}$. Since

$$f(z) = \frac{\sin z - z}{z^3} = \frac{(z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots) - z}{z^3} = -\frac{1}{3!} + \frac{1}{5!}z^2 - \cdots$$

it follows that $z = 0$ is a removable singularity of f.

(d) The function is not defined for $z_1 = 0$ and $z_2 = 1$. We have $f(z) = \frac{A(z)}{B(z)}$ for $A(z) = e^{1/z}$ and $B(z) = (z-1)^2$. Since A is holomorphic in $z_2 = 1$ and z_2 is a zero of multiplicity 2 for $B(z)$, it follows that f has a pole of order 2 in z_2.

We show that in $z_1 = 0$ f has an essential singularity.

To this end, we find the Laurent series expansion of f in $0 < |z| < 1$. Since

$$\frac{1}{(z-1)^2} = \Big(\sum_{k=0}^{\infty} z^k\Big)^2 = \sum_{k=0}^{\infty}\sum_{n=0}^{k} z^n z^{k-n} = \sum_{k=0}^{\infty}(k+1)z^k$$

we have

$$
\begin{aligned}
f(z) &= \Big(\sum_{k=0}^{\infty}(k+1)z^k\Big)\Big(\sum_{k=0}^{\infty}\frac{z^{-k}}{k!}\Big)\\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{k}(n+1)z^n\frac{z^{-(k-n)}}{(k-n)!}\\
&= \sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{n+1}{(k-n)!}z^{2n-k}.\\
&= \sum_{n=0}^{\infty}(n+1)[z^n+z^{n-1}+\frac{1}{2!}z^{n-2}+\frac{1}{3!}z^{n-3}+\cdots]\\
&= \big(1+z^{-1}+\frac{1}{2!}z^{-2}+\frac{1}{3!}z^{-3}+\cdots\big)+2\big(z+1+\frac{1}{2!}z^{-1}+\frac{1}{3!}z^{-2}+\cdots\big)\\
&\quad +3\big(z^2+z+\frac{1}{2!}+\frac{1}{3!}z^{-1}+\cdots\big)+\cdots
\end{aligned}
$$

Since there are infinitely many coefficients $a_{-n}\neq 0 (n\in\mathbb{N})$ in this representation, we can conclude that $z_1=0$ is an essential singularity of f.
The coefficient a_{-1} is the sum of the terms which stay in front of z^{-1}, i.e.

$$a_{-1}=\sum_{n=0}^{\infty}\frac{1^n}{n!}=e$$

Note: a_{-1} is can be calculated also from the Laurent series expansion of f for $|z|>1$. However, we made this choice of a_{-1} since it is exactly the residue of f in $z=0$.

Exercise 2

(a) Since z_0 is a pole of order $m\geq 1$ of f, we have

$$f(z)=\sum_{k=-m}^{\infty}a_k(z-z_0)^k.$$

Then

$$f'(z)=\sum_{k=-m}^{\infty}ka_k(z-z_0)^{k-1}$$

We get the coefficient in front of $(z-z_0)^{-1}$ for $k=0$. Therefore $Res(f';z_0)=0$.

(b) Since f' has a pole of order k, we can write it as

$$f'(z)=\frac{a_{-k}}{(z-z_0)^k}+\cdots\frac{a_{-1}}{z-z_0}+r(z)$$

with $r(z)$ holomorphic. Since the primitive of f' is holomorphic in $D\setminus\{z_0\}$, for every ball $B\subset D$ we have

$$0=\int_{\partial B}f'd\xi=\int_{\partial B}\frac{a_{-1}}{z-z_0}=2\pi i$$

Therefore $a_{-1}1=0$ and thus $k\geq 2$. The rest follows by integration.

Exercise 3 "(a) $\Rightarrow$ (c)": z_0 is a pole of order N for f, i.e.

$$f(z)=\frac{g(z)}{(z-z_0)^N},$$

where g is holomorphic in z_0 and $g(z_0)\neq 0$. Therefore, there exists a neighborhood $U_\rho(z_0)$ of z_0 in which g has no zeroes. Furthermore,

$$\frac{1}{f(z)}=\frac{(z-z_0)^N}{g(z)}$$

is holomorphic in z_0 and z_0 is a zero of order N for it.
"(c) ⇒ (a)": Since $1/f$ has a zero of order N in z_0, we can write it as

$$\frac{1}{f(z)} = (z - z_0)^N h(z)$$

with h holomorphic in z_0. Therefore

$$f(z) = \frac{1}{(z - z_0)^N h(z)} = \frac{g(z)}{(z - z_0)^N},$$

where $g := 1/h$ is holomorphic in z_0.
"(a) ⇒ (b)": Since g is holomorphic in z_0 it follows that g is bounded in a neighborhood $U_\rho(z_0)$ of z_0, i.e. there exist constants C_1, C_2 such that $C_1 \leq |g| \leq C_2$. Thus, for $f(z) = \frac{g(z)}{(z-z_0)^N}$, it holds

$$\frac{C_1}{|z - z_0|^N} \leq |f(z)| \leq \frac{C_2}{|z - z_0|^N}.$$

"(b) ⇒ (a)": Since we have $|(z - z_0)^N f(z)| \leq C_1$ for $z \in U_\rho(z_0) \setminus \{z_0\}$, it follows that the function $(z - z_0)^N f(z)$ has a removable singularity in z_0, i.e.

$$(z - z_0)^N f(z) = g(z)$$

with g holomorphic in z_0 and $g(z_0) \neq 0$.

Exercise 4 We always denote the integrands by f.

(a) f is holomorphic everywhere except in the points $z = \pm i$ - there it has poles of order 1. Since the positive oriented circle $|z| = 2$ goes once round these points, the residue theorem gives us

$$\int\limits_{|z|=2} f(z)\,dz = 2\pi i\big(Res(f; i) + Res(f; -i)\big).$$

For the residua we have

$$Res_{\pm i} f = \left[\frac{\cos z}{(z^2+1)'}\right]_{z=\pm i} = \left[\frac{\cos z}{2z}\right]_{z=\pm i} = \frac{\cos(\pm i)}{\pm 2i} = \pm\frac{\cos(i)}{2i}.$$

Since $Res(f; -i) = -Res(f; -i)$, we conclude that

$$\int\limits_{|z|=2} f(z)\,dz = 0..$$

(b) The zeroes of the denominator are $z = 2k\pi$ with $k \in \mathbb{Z}$. From them only $z = 0$ lies in the circle $|z| = 1$. Therefore

$$\int\limits_{|z|=1} f(z)\,dz = 2\pi i Res(f; 0).$$

Since

$$f(z) = \frac{z}{e^{iz} - 1} = \frac{z}{\big(1 + iz + \frac{1}{2}(iz)^2 + \cdots\big) - 1} = \frac{z}{iz - \frac{1}{2}z^2 + \cdots} = \frac{1}{i - \frac{1}{2}z + \cdots},$$

we see that f has a removable singularity in $z = 0$. Thus $Res(f; 0) = 0$ and therefore

$$\int\limits_{|z|=1} f(z)\,dz = 0.$$

(c) Here the residue theorem gives us

$$\int_{|z|=2} f(z)\,dz = 2\pi i Res(f; 1).$$

We consider the Laurent series expansion of f in order to compute the residue:

$$f(z) = \exp\left(\frac{z}{1-z}\right) = \exp\left(-1 + \frac{1}{1-z}\right) = e^{-1}e^{-1/(z-1)} = e^{-1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(z-1)^{-k}.$$

The coefficient in front of $\frac{1}{z-1}$ is $-e^{-1}$, i.e. $Res(f;1) = -e^{-1}$ and therefore

$$\int\limits_{|z|=2} \exp\left(\frac{z}{1-z}\right) dz = -\frac{2\pi i}{e}.$$

(d) First we find the zeroes of the denominator of f:

$$\begin{aligned}\cosh z = 1 &\iff e^z + e^{-z} = 2 \iff e^{2z} + 1 = 2e^z \\ &\iff (e^z)^2 - 2e^z + 1 = 0 \iff e^z = 1 \\ &\iff z = 2k\pi i \ \ (k \in \mathbb{Z})\end{aligned}$$

The boundary of G is given by

$$y^2 = (4\pi^2 - 1)(1 - x^2), \qquad \text{i.e.} \qquad x^2 + \frac{y^2}{4\pi^2 - 1} = 1,$$

which is an ellipse centered in 0 and with axes 2 and $2\sqrt{4\pi^2 - 1}$. Therefore, only the isolated singularity $z = 0$ lies in G and thus, by the residue theorem, we get

$$\int\limits_{\gamma} f(z)\, dz = 2\pi i Res(f;0).$$

Using the power series expansion of $coshz$ we get

$$f(z) = \frac{z}{\left(1 + \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \cdots\right) - 1} = \frac{1}{\frac{1}{2!}z + \frac{1}{4!}z^3 + \cdots} = \frac{\left(\frac{1}{2!} + \frac{1}{4!}z^2 + \cdots\right)^{-1}}{z},$$

i.e. f has a pole in $z = 0$. Therefore

$$Res(f;0) = \lim_{z\to 0} zf(z) = \lim_{z\to 0}\frac{1}{\frac{1}{2!} + \frac{1}{4!}z^2 + \cdots} = 2,$$

and thus

$$\int\limits_{\gamma} f(z)\, dz = 4\pi i.$$