Karlsruher Institut für Technologie
Institut für Analysis
Dr. A. Müller-Rettkowski
M. Sc. S. Avramska

SS 2012
20.07.2012

Funktionentheorie
Exercise Sheet 14
Solutions

Exercise 1

(a) Obviously we have

$$\int_0^\infty \frac{t^2+1}{t^4+1}\,dt = \frac{1}{2}\int_{-\infty}^\infty \frac{t^2+1}{t^4+1}\,dt.$$

Moreover, the denominator of the function $f = P/Q$ with $P(x) := x^2+1$ and $Q(x) := x^4+1$ $(x \in \mathbb{R})$ has no real zeroes. Furthermore, we have

$$degQ - degP = 4 - 2 \geq 2$$

and therefore, by Theorem 2 from Chapter 17.2, we get

$$\int_{-\infty}^\infty \frac{t^2+1}{t^4+1}\,dt = 2\pi i(Res(f, z_1) + Res(f, z_2)),$$

where $z_1 = \frac{\sqrt{2}}{2}(1+i)$ and $z_2 = \frac{\sqrt{2}}{2}(-1+i)$ (the poles of f laying in the upper half-plane).
We calculate

$$Res(f; z_k) = \frac{P(z_k)}{Q'(z_k)} = \frac{z_k^2+1}{4z_k^3} = -\frac{\sqrt{2}}{4}i \quad (k = 1, 2)$$

and thus

$$\int_0^\infty \frac{t^2+1}{t^4+1}\,dt = -\frac{1}{2}\cdot 2\pi i \cdot 2\frac{\sqrt{2}}{4}i = \frac{\sqrt{2}}{2}\pi$$

(b) We have $f(z) = e^{i\alpha z}\frac{P(z)}{Q(z)}$ for $z \in \mathbb{C}\setminus N(Q)$ and $\alpha := \pi, P := 1, Q(x) := x^2+2x+2 = (x+1)^2+1$ $(x \in \mathbb{R})$.
Obviously Q has no real zeroes. Moreover

$$degQ - degP = 2 - 0 \geq 1$$

and thus, using Theorem 3 from Chapter 17.3, we can conclude

$$\int_{-\infty}^\infty \frac{e^{i\pi t}}{t^2+2t+2}\,dt = 2\pi i Res(f, z_1),$$

where $z_1 = -1+i$.
We calculate

$$Res(f; z_1) = \frac{e^{i\pi z_1}P(z_1)}{Q'(a_1)} = \frac{e^{-i\pi-\pi}}{-2+2i+2} = \frac{-e^{-\pi}}{2i}$$

and therefore

$$\int_{-\infty}^\infty \frac{e^{i\pi t}}{t^2+2t+2}\,dt = 2\pi i\frac{-e^{-\pi}}{2i} = -\pi e^{-\pi}.$$

(c) Obviously we have

$$\int_0^\infty \frac{\cos t}{t^2+a^2}dt = \frac{1}{2}Re\Big(\int_{-\infty}^\infty \frac{e^{it}}{t^2+a^2}dt\Big).$$

Since the only pole of $f(z) = \frac{e^{it}}{t^2+a^2}$ in the upper half-plane is $z_1 = ia$, by Theorem 3 from Chapter 17.3, we get

$$\int_0^\infty \frac{\cos t}{t^2+a^2}dt = \frac{1}{2}Re\big(2\pi i Res(f;a_1)\big).$$

We calculate

$$Res(f;a_1) = \frac{e^{-a}}{2ai}$$

and thus

$$\int_0^\infty \frac{\cos t}{t^2+a^2}dt = \frac{\pi}{2a}e^{-a}.$$

(d) First we note that $\frac{t\cos t}{t^2-2\pi t+a} = Re\Big(\frac{te^{it}}{t^2-2\pi t+a}\Big)$ for $t \in \mathbb{R}$. Moreover, we have $f(z) = e^{i\alpha z}\frac{P(z)}{Q(z)}$ for $b := a-\pi^2 > 0, \alpha := 1, P(x) := x, Q(x) := x^2 - 2\pi x + a = (x-\pi)^2 + b \ \ (x \in \mathbb{R})$. Since $b > 0$, the function Q has no real zeroes. Moreover, we have

$$degQ - degP = 2 - 1 \geq 1$$

and thus, by Theorem 3 from Chapter 17.3, we get

$$\int_{-\infty}^\infty \frac{te^{it}}{t^2-2\pi t+a}\,dt = 2\pi i Res(f,z_1)$$

with $z_1 = \pi + i\sqrt{b}$.
We calculate

$$Res(f;z_1) = \frac{e^{iz_1}P(z_1)}{Q'(z_1)} = \frac{e^{i\pi-\sqrt{b}}(\pi+i\sqrt{b})}{2(\pi+i\sqrt{b})-2\pi} = \frac{-e^{-\sqrt{b}}(\pi+i\sqrt{b})}{2i\sqrt{b}}$$

and therefore

$$\int_{-\infty}^\infty \frac{t\cos t}{t^2-2t+a}\,dt = \mathrm{Re}(2\pi i\frac{-e^{-\sqrt{b}}(\pi+i\sqrt{b})}{2i\sqrt{b}} = \frac{-\pi^2 e^{-\sqrt{b}}}{\sqrt{b}} = \frac{-\pi^2 e^{-\sqrt{a-\pi^2}}}{\sqrt{a-\pi^2}}.$$

Exercise 2 As suggested in the hint, we will integrate over the given contour γ. We use the following parametrization

$$\begin{aligned} \gamma_1 &:= r, \quad r \in [0,R] \\ \gamma_2 &:= Re^{i\theta}, \quad \theta \in [0,\frac{2\pi}{n}] \\ \gamma_3 &:= re^{i\frac{2\pi}{n}}, \quad r \in [R,0] \end{aligned}$$

The only zero of the denominator of the integrand which is in the interior of the domain enclosed by γ is $e^{\frac{i\pi}{n}}$ and thus, by the Residue theorem, we have

$$\int_\gamma \frac{1}{1+z^n}dz = 2\pi i Res\Big(\frac{1}{1+z^n};e^{\frac{i\pi}{n}}\Big). \tag{1}$$

First we calculate

$$\begin{aligned}\int_{\gamma} \frac{1}{1+z^n} dz &= \int_{\gamma_1} \frac{1}{1+z^n} dz + \int_{\gamma_2} \frac{1}{1+z^n} dz + \int_{\gamma_3} \frac{1}{1+z^n} dz \\ &= \int_0^R \frac{1}{1+r^n} dr + \int_0^{2\pi/n} \frac{1}{1+R^n e^{in\theta}} Rie^{i\theta} d\theta + \int_R^0 \frac{1}{1+r^n} e^{i\frac{2\pi}{n}} dr \\ &= \left(1 - e^{i\frac{2\pi}{n}}\right) \int_0^R \frac{1}{1+r^n} dr + i\frac{R}{R^n} \int_0^{2\pi/n} \frac{1}{\frac{1}{R^n} + e^{in\theta}} e^{i\theta} d\theta.\end{aligned}$$

Now we compute

$$Res\Big(\frac{1}{1+z^n}; e^{\frac{i\pi}{n}}\Big) = \lim_{z \to e^{i\pi n}} (z - e^{i\pi n}) \frac{1}{1+z^n} = -\frac{e^{\frac{i\pi}{n}}}{n}.$$

Thus, letting $R \to \infty$ in (1), we get

$$\int_{\gamma} \frac{1}{1+z^n} dz = -2\pi i \frac{e^{\frac{i\pi}{n}}}{n} \left(1 - e^{i\frac{2\pi}{n}}\right)^{-1} = \frac{\pi}{n} \frac{1}{\sin\left(\frac{\pi}{n}\right)}.$$