Karlsruher Institut für Technologie
Institut für Analysis
Dr. A. Müller-Rettkowski
M. Sc. S. Avramska

SS 2012
30.04.2012

Funktionentheorie
Exercise Sheet 2
Solutions

Exercise 1

(a) It holds:

$$\begin{aligned}
|z-a| < |1-\bar{a}z| &\iff |z-a|^2 < |1-\bar{a}z|^2 \\
&\iff (z-a)(\bar{z}-\bar{a}) < (1-\bar{a}z)(1-a\bar{z}) \\
&\iff z\bar{z} - a\bar{z} - z\bar{a} + a\bar{a} < 1 - a\bar{z} - z\bar{a} + a\bar{a}z\bar{z} \\
&\iff |z|^2 + |a|^2 < 1 + |a|^2|z|^2 \\
&\iff |z|^2(1-|a|^2) < 1 - |a|^2 \\
&\overset{|a|<1}{\iff} |z|^2 < 1 \\
&\iff |z| < 1
\end{aligned}$$

(b) Note: due to $|a| < 1, |z| \leq 1$, it follows that $1 - \bar{a}z \neq 0$. It holds:

$$\begin{aligned}
\left(\frac{\big||z|-|a|\big|}{1-|az|}\right)^2 &= \frac{(|z|-|a|)^2}{(1-|az|)^2} \\
&= \frac{|z|^2 - 2|a||z| + |a|^2}{(1-|az|)^2} \\
&= 1 - \frac{1 + |a|^2|z|^2 - |z|^2 - |a|^2}{(1-|az|)^2}.
\end{aligned}$$

Since $|\bar{a}| = |a| < 1$ and $|z| \leq 1$, it follows that $1 + |a|^2|z|^2 - |z|^2 - |a|^2 = (1-|z|^2)(1-|a|^2) \geq 0$ and $0 < 1 - |az| = 1 - |\bar{a}z| = \big|1 - |\bar{a}z|\big| \leq |1 - \bar{a}z|$. It follows

$$\begin{aligned}
1 - \frac{1 + |a|^2|z|^2 - |z|^2 - |a|^2}{(1-|az|)^2} &\leq 1 - \frac{1 + |a|^2|z|^2 - |z|^2 - |a|^2}{|1-\bar{a}z|^2} \\
&= \frac{|1-\bar{a}z|^2 - 1 - |a|^2|z|^2 + |z|^2 + |a|^2}{|1-\bar{a}z|^2} \\
&= \frac{1 - \bar{a}z - a\bar{z} + |a|^2|z|^2 - 1 - |a|^2|z|^2 + |z|^2 + |a|^2}{|1-\bar{a}z|^2} \\
&= \frac{-\bar{a}z - a\bar{z} + |z|^2 + |a|^2}{|1-\bar{a}z|^2} \\
&= \frac{(z-a)^2}{|1-\bar{a}z|^2} \\
&= \left|\frac{z-a}{1-\bar{a}z}\right|^2.
\end{aligned}$$

Thus, it follows

$$\frac{\big||z|-|a|\big|}{1-|az|} \leq \left|\frac{z-a}{1-\bar{a}z}\right|.$$

Exercise 2

(a) f_1 is not continuous in 0:
For $n \in \mathbb{N}$ let $z_n = \frac{i}{n}$. Then $z_n \xrightarrow{n\to\infty} 0$, but $f_1(z_n) \nrightarrow 0 = f_1(0)$ for $n \to \infty$, because

$$f_1(z_n) = \frac{\frac{1}{n}}{\frac{i}{n}} = \frac{1}{i} = -i \quad \text{for all } n \in \mathbb{N}.$$

f_2 is continuous in 0:
For every $z \in \mathbb{C} \setminus \{0\}$ it holds

$$|f_2(z)| = |z^{-1}||\mathrm{Im} z^2| \leq |z^{-1}||z^2| = |z| \xrightarrow{z\to 0} 0.$$

Therefore $f_2(z) \xrightarrow{z\to 0} 0 = f_2(0)$.
f_3 is continuous in 0:
For every $z \in \mathbb{C} \setminus \{0\}$ it holds

$$|f_3(z)| = |z^{-1}||\mathrm{Im} z^2|^2 \leq |z^{-1}||z^2|^2 = |z|^3 \xrightarrow{z\to 0} 0.$$

Therefore $f_3(z) \xrightarrow{z\to 0} 0 = f_3(0)$.
(The following shows that the function $\tilde{f}_2 = z^{-2}Im(z^2)$ is not continuous in 0 and the function $\tilde{f}_3 = z^{-2}(Im(z^2))^2$ is continuous in 0:
$\tilde{f}_2$ is not continuous in 0:
For $n \in \mathbb{N}$ let $z_n = \frac{i+1}{n}$. Then $z_n \xrightarrow{n\to\infty} 0$. Furhtermore, $z_n^2 = \frac{(1+i)^2}{n^2} = \frac{2i}{n^2}$ and therefore

$$\tilde{f}_2(z_n) = \frac{\frac{2}{n^2}}{\frac{2i}{n^2}} = -i \quad \text{for all } n \in \mathbb{N}.$$

Thus $\tilde{f}_2(z_n) \nrightarrow 0 = \tilde{f}_2(0)$ for $n \to \infty$.
$\tilde{f}_3$ is continuous in 0:
For every $z \in \mathbb{C} \setminus \{0\}$ it holds

$$|\tilde{f}_3(z)| = |z^{-2}||\mathrm{Im} z^2|^2 \leq |z^{-2}||z^2|^2 = |z|^2 \xrightarrow{z\to 0} 0.$$

Therefore $\tilde{f}_3(z) \xrightarrow{z\to 0} 0 = \tilde{f}_3(0)$.)

(b) If both of the properties were true, then it would hold

$$1 = (f(1))^2 = f(1)f(1) = f(1.1) = f(1),$$
$$-1 = (f(-1))^2 = f(-1)f(-1) = f((-1)(-1)) = f(1),$$

which is a contradiction.

(c) Let $a \in \mathbb{C} \setminus \{0\}$. Define the function

$$g(z) := \frac{f(a)f(z)}{f(az)}, \quad z \in \mathbb{C} \setminus \{0\}.$$

It takes only the values ± 1. Since g is continuous, it follows that it is constant and the value of this constant is either 1 or -1. Assume that $g(z) \equiv 1$. Then we have

$$f(a)f(z) = f(az)$$

and we can use part (b).
Analogously, if we assume that $g(z) \equiv -1$, it would follow that

$$f(a)f(z) = -f(az)$$

and similarly to part (b), we would have

$$1 = (f(1))^2 = f(1)f(1) = -f(1.1) = -f(1),$$
$$-1 = (f(-1))^2 = f(-1)f(-1) = -f((-1)(-1)) = -f(1),$$

which is a contradiction again.

Exercise 3 We know that

$$\varepsilon|z|^2 + \bar{\alpha}z + \alpha\bar{z} + \beta = 0, \quad z, \alpha \in \mathbb{C}, \beta \in \mathbb{R} \tag{1}$$

defines a circle (for $\varepsilon = 1$ and $\beta < |\alpha|^2$) or a line (for $\varepsilon = 0$) in $\mathbb{C}$.
Furthermore, we know that a circle on Σ is defined by

$$(*) \qquad Ax_1 + Bx_2 + Cx_3 + D = 0 \quad \text{for } A, B, C, D \in \mathbb{R} \text{ and } x_1^2 + x_2^2 + x_3^2 = 1.$$

(a) By the definition of $p = \pi^{-1}$ it follows that for $\mathbb{C} \ni z = x + iy$ we have $p(z) = \frac{1}{x^2+y^2+1}(2x, 2y, x^2 + y^2 - 1)$. Plugging this in $(*)$, we get

$$\frac{2Ax}{x^2 + y^2 + 1} + \frac{2By}{x^2 + y^2 + 1} + \frac{C(x^2 + y^2 - 1)}{x^2 + y^2 + 1} + D = 0,$$

which is equivalent to

$$(C + D)(x^2 + y^2) + 2Ax + 2By + (D - C) = 0$$

and this is similar to (1).

(b) Let $\alpha = a + ib$. Furthermore, let $Z = (x_1, x_2, x_3) \in \Sigma$ (i.e., $x_1^2 + x_2^2 + x_3^2 = 1$) and $z \in \mathbb{C}$ be the projection of Z on $\mathbb{C}$, i.e., $z = \pi(x_1, x_2, x_3) = \frac{x_1}{1-x_3} + i\frac{x_2}{1-x_3}$. Then $|z|^2 = \frac{1+x_3}{1-x_3}$.
Therefore, by (1), we have

$$\varepsilon(1 + x_3) + 2ax_1 + 2bx_2 + \beta(1 - x_3) = 0 \iff 2ax_1 + 2bx_2 + (\varepsilon - \beta)x_3 = -\beta - \varepsilon,$$

which defines a level on Σ, i.e. a circle.

Exercise 4

(a) $d(z, w) \geq 0$ for every $z, w \in \hat{\mathbb{C}}$ and $d(z, w) = 0 \iff |z - w| = 0$, i.e., $z = w$.

(b) $d(z, w) = d(w, z)$ for every $z, w \in \hat{\mathbb{C}}$.

(c) We will show that

$$d(z, w) := 2\frac{|z - w|}{\sqrt{|z|^2 + 1}\sqrt{|w|^2 + 1}} = \|p(z) - p(w)\| =: \chi(z, w) \quad z, w \in \hat{\mathbb{C}}$$

We have

$$\begin{aligned}
\|p(z) - p(w)\|^2 &= \sum_{k=1}^{n}(x_k - \eta_k)^2 \\
&= \sum_{k=1}^{n} x_k^2 + \sum_{k=1}^{n} \eta_k^2 - 2\sum_{k=1}^{n} x_k\eta_k \\
&= 2 - 2\frac{(z + \bar{z})(w + \bar{w}) + i^2(z - \bar{z})(w + \bar{w}) + (|z|^2 - 1)(|w|^2 - 1)}{(|z|^2 + 1)(|w|^2 + 1)} \\
&= 4\frac{|w|^2 + |z|^2 - z\bar{w} - \bar{z}w}{(|z|^2 + 1)(|w|^2 + 1)} \\
&= 4\frac{|z - w|^2}{(|z|^2 + 1)(|w|^2 + 1)},
\end{aligned}$$

i.e.,

$$d(z, w) = \|p(z) - p(w)\|.$$

Note that

$$\begin{aligned}
d(z, \infty) &= \|p(z) - p(\infty)\| \\
&= \sqrt{x_1^2 + x_2^2 + (x_3 - 1)^2} \\
&= \frac{2}{\sqrt{|z|^2 + 1}}.
\end{aligned}$$

Thus, we have

$$d(z, w) = \|p(z) - p(w)\| \leq \|p(z) - p(v)\| + \|p(v) - p(w)\| = d(z, v) + d(v, w) \quad \text{for all } z, v, w \in \hat{\mathbb{C}}.$$