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Funktionentheorie
Exercise Sheet 3
Solutions

Exercise 1

(a) $u(x,y) = x^4 - 6x^2y^2 + y^4 - x^2 + y^2$; $u_x(x,y) = 4x^3 - 12xy^2 - 2x$; $u_y(x,y) = -12x^2y + 4y^3 + 2y$;
$v(x,y) = 4x^3y - 4xy^3 - 2xy$; $v_x(x,y) = 12x^2y - 4y^3 - 2y$; $v_y(x,y) = 4x^3 - 12xy^2 - 2x$;

I.e. $u_x = v_y$ and $u_y = -v_x$ for all $x, y, \in \mathbb{R}$. Thus f is differetiable in all $x, y, \in \mathbb{R}$ and

$$f'(x+iy) = u_x(x,y) + iv_x(x,y) = 4x^3 - 12xy^2 - 2x + i(12x^2y - 4y^3 - 2y).$$

(b) $u(x,y) = \cosh(x)\cos(y)$; $u_x(x,y) = \sinh(x)\cos(y)$; $u_y(x,y) = -\cosh(x)\sin(y)$;
$v(x,y) = \sinh(x)\sin(y)$; $v_x(x,y) = \cosh(x)\sin(y)$; $v_y(x,y) = \sinh(x)\cos(y)$;

I.e. $u_x = v_y$ and $u_y = -v_x$ for all $x, y, \in \mathbb{R}$. Thus f is differetiable in all $x, y, \in \mathbb{R}$ and

$$f'(x+iy) = u_x(x,y) + iv_x(x,y) = \sinh(x)\cos(y) + \cosh(x)\sin(y).$$

(c) $u(x,y) = x^3 - 2xy^2$; $u_x(x,y) = 3x^2 - 2y^2$; $u_y(x,y) = -4xy$;
$v(x,y) = 2x^2y - y^3$; $v_x(x,y) = 4xy$; $v_y(x,y) = 2x^2 - 3y^2$;

I.e. $u_y = -v_x$ for all $x, y, \in \mathbb{R}$ and $u_x = v_y$ only for $x = y = 0$. Thus f is differetiable only in 0 and

$$f'(0) = u_x(0,0) + iv_x(0,0) = 0.$$

(d) $u(x,y) = \sin(x)\sin(y)$; $u_x(x,y) = \cos(x)\sin(y)$; $u_y(x,y) = \sin(x)\cos(y)$;
$v(x,y) = -\cos(x)\cos(y)$; $v_x(x,y) = \sin(x)\cos(y)$; $v_y(x,y) = \cos(x)\sin(y)$;

I.e. $u_x = v_y$ for all $x, y, \in \mathbb{R}$ and $u_y = -v_x$ only for $\sin(x) = 0$ or $\cos(y) = 0$, i.e., for $x = \pi\mathbb{Z}$ or $y = \frac{\pi}{2}\mathbb{Z}$. Thus f is differetiable only in the points

$$(x,y) \in \big((\pi\mathbb{Z}, \mathbb{R}) \cup (\mathbb{R}, \frac{\pi}{2} + \pi\mathbb{Z})\big) =: D$$

and

$$f'(x+iy) = u_x(x,y) + iv_x(x,y) = \cos(x)\sin(y) + \underbrace{\sin(x)cos(y)}_{=0} = \cos(x)\sin(y) \quad \text{for all } (x,y) \in D.$$

Exercise 2

(a) Assume that there is $z \in D^\circ$. Then there exists $\delta > 0$ such that $D(z,\delta) \subset D$. Then for all $\varepsilon > 0$ we have $\big(D(z,\delta)\backslash\{z\}\big) \cap D \supset D(z, \min\{\delta,\varepsilon\})\backslash\{z\} \neq \emptyset$, i.e. z is an accumulation point of $D \subset M$, which is a contradiction.

(b) For every $z \in M$ there exists $\varepsilon_z > 0$ such that $\big(D(z,\varepsilon_z)\backslash\{z\}\big) \cap D = \emptyset$. For every $z, v \in D$ with $z \neq v$ it holds

$$D(z, \frac{\varepsilon_z}{2}) \cap D(v, \frac{\varepsilon_v}{2}) = \emptyset \tag{1}$$

(otherwise there would exists $s \in D(z, \frac{\varepsilon_z}{2}) \cap D(v, \frac{\varepsilon_v}{2})$ and it would hold

$$|z - v| \leq |z - s| + |s - v| \leq \frac{\varepsilon_z}{2} + \frac{\varepsilon_v}{2} \leq \max\{\varepsilon_z, \varepsilon_v\}. \tag{2}$$

Without loss of generality, we may assume that $\varepsilon_z \geq \varepsilon_v$. Then, by (2), it follows that $v \in \big(D(z,\varepsilon_z)\backslash\{z\}\big) \cap D = \emptyset$, which is a contradiction). For every $z \in D$ there exists $q_z \in \big(D(z, \frac{\varepsilon_z}{2}) \cap (\mathbb{Q} + i\mathbb{Q})\big)$. By (1), $z \mapsto q_z$ is injective, and since $\mathbb{Q} + i\mathbb{Q}$ is countable, D is also countable.

(c) "$\Rightarrow$": clear. "$\Leftarrow$": Assume that D is infinite. Then there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ with different members. Since D is compact, there exists a subsequence $\{z_{n_k}\}_{n_k\in\mathbb{N}}$ of $\{z_n\}_{n\in\mathbb{N}}$ such that $z_{n_k} \to z_0 \in D$. Furthermore, almost all $z_{n_k} \neq z_0$, i.e., z_0 is an accumulation point of D in M, i.e. D is not discrete in M, which is a contradiction.

(d) Assume that D is not discrete in itself. Then there exists $z \in D$ such that z is an accumulation point of D. Since $D \subset M$, it follows that z is an accumulation point of D in M, i.e. D is not discrete in M, which is a contradiction.
Consider the space $D := \{\frac{1}{n} : n \in \mathbb{N}\}$. Then D is discrete in itself but not in $\mathbb{C}$, since its accumulation point is $0 \in \mathbb{C}$.

Exercise 3

(a) For $a \in A$ let $f : \mathbb{C} \to \mathbb{R}$ be defined as $f(z) := |z - a|$. Then we have

$$f(z_1) = |z_1 - a| \leq |z_1 - z_2| + |z_2 - a| = |z_1 - z_2| + f(z_2), \quad \text{i.e. } f(z_1) - f(z_2) \leq |z_1 - z_2|$$
$$f(z_2) = |z_2 - a| \leq |z_2 - z_1| + |z_1 - a| = |z_2 - z_1| + f(z_1), \quad \text{i.e. } f(z_2) - f(z_1) \leq |z_1 - z_2|$$

Thus

$$|f(z_1) - f(z_2)| \leq |z_1 - z_2| \to 0 \quad \text{as } z_1 \to z_2,$$

i.e. f is continuous.

(b) By the definition of $dist(K, A)$, there exists a sequence $\{(z_n, w_n)\}_{n\in\mathbb{N}} \in K \times A$ sucht that $(z_n, w_n) \to r := dist(K, A)$. Since K is compact, the sequence $\{z_n\}_{n\in\mathbb{N}} \in K$ has a subsequence $\{z_{n_1}\}_{n\in\mathbb{N}}$ such stat $z_{n_1} \to z_0 \in K$. Then

$$|w_{n_1} - z_0| \leq |w_{n_1} - z_{n_1}| + |z_{n_1} - z_0| \to r,$$

i.e. the sequence $\{w_{n_1}\}_{n\in\mathbb{N}}$ is bounded. By Bolzano-Weierstrass Theorem, there exists a subsequence $\{w_{n_2}\}_{n\in\mathbb{N}}$ of $\{w_{n_1}\}_{n\in\mathbb{N}}$ such that $w_{n_2} \to w_0$. Since A is closed, it follows that $w_0 \in A$. Therefore

$$r \leftarrow |w_{n_2} - z_{n_2}| \to |w_0 - z_0|, \quad \text{i.e. } |w_0 - z_0| = dist(K, A)$$

Exercise 4

(a) We want to show that $x \in f^{-1}(U \cup V) \iff x \in f^{-1}(U) \cup f^{-1}(V)$.

$$\begin{aligned} x \in f^{-1}(U \cup V) &\iff f(x) \in (U \cup V) \\ &\iff f(x) \in U \text{ or } f(x) \in V \\ &\iff x \in f^{-1}(U) \text{ or } x \in f^{-1}(V) \\ &\iff x \in f^{-1}(U) \cup f^{-1}(V). \end{aligned}$$

(b) We want to show that $x \in f^{-1}(U \cap V) \iff x \in f^{-1}(U) \cap f^{-1}(V)$.

$$\begin{aligned} x \in f^{-1}(U \cap V) &\iff f(x) \in (U \cap V) \\ &\iff f(x) \in U \text{ and } f(x) \in V \\ &\iff x \in f^{-1}(U) \text{ and } x \in f^{-1}(V) \\ &\iff x \in f^{-1}(U) \cap f^{-1}(V). \end{aligned}$$

(c) We want to show that $x \in f^{-1}(U \setminus V) \iff x \in f^{-1}(U) \setminus f^{-1}(V)$.

$$\begin{aligned} x \in f^{-1}(U \setminus V) &\iff f(x) \in (U \setminus V) \\ &\iff f(x) \in U \text{ and } f(x) \notin V \\ &\iff x \in f^{-1}(U) \text{ and } x \notin f^{-1}(V) \\ &\iff x \in f^{-1}(U) \setminus f^{-1}(V). \end{aligned}$$