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Funktionentheorie
Exercise Sheet 4
Solutions

Exercise 1 For $z = x + iy$, we have

$$F(x,y) := f(x+iy) = e^{-\frac{1}{(x+iy)^4}} \quad \text{for } x \neq 0, y \neq 0, \tag{1}$$

and thus

$$D_1F(x,y) = 4e^{-\frac{1}{(x+iy)^4}}\frac{1}{(x+iy)^5},$$
$$D_2F(x,y) = 4ie^{-\frac{1}{(x+iy)^4}}\frac{1}{(x+iy)^5},$$

i.e., $D_2F(x,y) = iD_1F(x,y)$ for all $x \neq 0, y \neq 0$. Clearly, for $x = 0, y = 0$, f fulfills the Cauchy-Riemann conditions.

However, near the origin f is unbounded (consider it $z = \varepsilon(1+i)$), so it cannot be analytic in the whole complex plane.

Exercise 2

(a)

$$u_x(x,y) = 3x^2 - 3y^2$$

From $u_x = v_y$ it follows

$$v(x,y) = \int (3x^2 - 3y^2)dy = 3x^2y - y^3 + c(x).$$

From $u_y = -v_x$ it follows that

$$c'(x) = 0, \quad \text{i.e. } c(x) = const := 0.$$

Therefore, the function we are looking for is

$$f(x+iy) = x^3 - 3xy^2 + 1 + i(3x^2y - y^3) = (x+iy)^3 + 1,$$
$$f(z) = z^3 + 1 \quad \text{for } z = x + iy.$$

(b)

$$u_y(x,y) = \frac{-2xy}{(x^2+y^2)^2} \text{for all } x \neq 0, y \neq 0$$

From $u_y = -v_x$ it follows

$$v(x,y) = \int \frac{2xy}{(x^2+y^2)^2}dx = \frac{-y}{x^2+y^2} + c(y).$$

From $u_x = v_y$ it follows

$$c'(y) = 0 \quad \text{i.e. } c(y) = const := 0.$$

Therefore, the function we are looking for is

$$f(x+iy) = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2},$$
$$f(z) = \frac{1}{z} \quad \text{for } z = x + iy.$$

(c)

$$v_x(x,y) = e^x(x\sin(y) + y\cos(y) + \sin(y)$$

From $v_x = -u_y$ it follows

$$u(x,y) = -\int e^x(x\sin(y) + y\cos(y) + \sin(y)dx = e^x(x\cos(y) - y\sin(y)) + c(x).$$

From $v_y = u_x$ it follows that

$$c'(x) = 0 \quad \text{i.e. } c(x) = const := 0.$$

Therefore, the function we are looking for is

$$f(x+iy) = e^x(x\cos(y) - y\sin(y) + i(x\sin(y) + y\cos(y)) = e^{x+iy}(x+iy),$$
$$f(z) = ze^z \quad \text{for } z = x+iy.$$

Exercise 3

(a) E.g. $c = i$. Then we have

$$a_0^{(i)} = 0, \quad a_1^{(i)} = i, \quad a_2^{(i)} = -1+i, \quad a_3^{(i)} = -i \quad a_4^{(i)} = a_2^{(i)}, \quad a_5^{(i)} = a_3^{(i)}, \dots,$$

so the sequence $\{a_n^{(i)}\}$ is divergent.

(b) First we note that if there is $\mu \in \mathbb{N}$ such that $|a_\mu^{(c)}| \geq \max\{|c|, 2\}$ then for all $n \in \mathbb{N}$ with $n \geq \mu$ we have $|a_n^{(c)}| \geq \max\{|c|, 2\}$:

$$|a_{\mu+1}^{(c)}| = |(a_\mu^{(c)})^2 + c| \geq |a_\mu^{(c)}|^2 - |c| \geq (|a_\mu^{(c)}| - 1)|c| \geq |c|,$$
$$|a_{\mu+1}^{(c)}| = |(a_\mu^{(c)})^2 + c| \geq |a_\mu^{(c)}|^2 - |c| \geq (|a_\mu^{(c)}| - 1)|a_\mu^{(c)}| \geq |a_\mu^{(c)}| \geq 2.$$

Therefore, for all $n \in \mathbb{N}$ with $n \geq \mu$ it holds

$$|a_n^{(c)}| \geq (|a_\mu^{(c)}| - 1)^{n-\mu}|a_\mu^{(c)}| \tag{2}$$

Let $N \in \mathbb{N}$ be such that $|a_N^{(c)}| > 2$. We consider the following two cases:
Case 1: $|c| > 2$. By $|a_1^{(c)}| = |c| \geq \max\{|c|, 2\}$, we get that μ from above is 1 and by (2) it follows

$$|a_n^{(c)}| \geq (|a_1^{(c)}| - 1)^{n-1}|a_1^{(c)}| \to \infty \quad \text{as } n \to \infty,$$

Case 2: $|c| \leq 2$. Since $|a_N^{(c)}| > 2$ it follows that $|a_N^{(c)}| \geq \max\{|c|, 2\}$ and by (2) we get

$$|a_n^{(c)}| \geq (|a_N^{(c)}| - 1)^{n-N}|a_N^{(c)}| \to \infty \quad \text{as } n \to \infty,$$

Therefore, $\{a_n^{(c)}\}$ is unbounded.

(c) We will show that M is bounded and closed:

- <u>bounded:</u> Assume that there is $c \in M$ with $|c| > 2$. Then $|a_1^{(c)}| = |c| > 2$ and by part (b) we can conclude that $\{a_n^{(c)}\}$ is unbounded, which is a contradiction. Thus $|c| \leq 2$ for all $c \in M$.
- <u>closed:</u> For every $c \in \mathbb{C}$ we have either $|a_n^{(c)}| \leq 2$ for all $n \in \mathbb{N}$ (and so $c \in M$) or there exists $n \in \mathbb{N}$ with $|a_n^{(c)}| > 2$ (so by part (b) it follows that $\{a_n^{(c)}\}$ is unbounded, i.e. $c \notin M$). Therefore

$$\mathbb{C} \setminus M = \bigcup_{n \in \mathbb{N}} \{c \in \mathbb{C} : |a_n^{(c)}| > 2\}.$$

  By induction, it can be shown that $c \mapsto a_n^{(c)}$ is a polynomial over $\mathbb{C}$ and then, by the hint, we can conclude that $\mathbb{C} \setminus M = \bigcup_{n \in \mathbb{N}} \{c \in \mathbb{C} : |a_n^{(c)}| > 2\}$ is open. Thus M is closed.

(d) By part (c): $|c| \leq 2$ for all $c \in M$. Consider $\{a_n^{(c)}\}$ for $c = -2$. Then $\{a_n^{(-2)}\} = (0, -2, 2, 2, 2, \dots)$, and so $-2 \in M$. Thus $max\{|c| : c \in M\} = 2$.

Exercise 4 From the convergence of the series $\sum\limits_{k=1}^{\infty} z_k$ follows the convergence of $\sum\limits_{k=1}^{\infty} Rez_k$, which means that $Rez_k \to 0$ as $k \to \infty$. Since we have $Rez_k > 0$ it follows that $Rez_k \in [0,1]$ for all large enough $k \in \mathbb{N}$. Therefore we have

$$0 \leq (Rez_k)^2 \leq Rez_k.$$

By the Majorant criterion, we can conclude that $\sum\limits_{k=1}^{\infty} (Rez_k)^2$ converges.

From the convergence of $\sum\limits_{k=1}^{\infty} z_k^2$ follows the convergence of

$$\begin{aligned}\sum_{k=1}^{\infty} Re(z_k)^2 &= \sum_{k=1}^{\infty} Re(x_k^2 + 2ix_k y_k + i^2 y_k) \\ &= \sum_{k=1}^{\infty} (x_k^2 - y_k^2) \\ &= \sum_{k=1}^{\infty} \left((Rez_k)^2 - (Imz_k)^2\right)\end{aligned}$$

and therefore we conclude the convergence of

$$2\sum_{k=1}^{\infty} (Rez_k)^2 - \sum_{k=1}^{\infty} \left((Rez_k)^2 - (Imz_k)^2\right) = \sum_{k=1}^{\infty} \left((Rez_k)^2 + (Imz_k)^2\right) = \sum_{k=1}^{\infty} |z_k|^2.$$

Counterexample: Consider $\{z_k := \frac{i(-1)^k}{k}\}_{k \in \mathbb{N}}$. Then it holds $Rez_k = 0 \geq 0$. Furthermore $\sum\limits_{k=1}^{\infty} z_k = i \sum\limits_{k=1}^{\infty} \frac{(-1)^k}{k}$ and $\sum\limits_{k=1}^{\infty} z_k^2 = -\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ are convergent. But $\sum\limits_{k=1}^{\infty} |z_k| = \sum\limits_{k=1}^{\infty} \frac{1}{k}$ is divergent.