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Funktionentheorie
Exercise Sheet 5
Solutions

Exercise 1

(a) (1) Let $a_n := \frac{5+i}{4-2i}$. Since it holds $|5+i| = \sqrt{26} > \sqrt{20} = |4-2i|$ we have $|a_n| > 1$ for all $n \in \mathbb{N}$. Therefore, $a_n \nrightarrow 0$ for $n \to \infty$ and thus the series is divergent.

(2) Let $b_n := \left(\frac{1}{2i}\right)^n \left(1 + \frac{i}{n}\right)^{n^2}$. Then

$$|b_n| = \left|\frac{1}{2i}\right|^n \left|1 + \frac{i}{n}\right|^{n^2} = \frac{1}{2^n}\left(1 + \frac{1}{n^2}\right)^{\frac{n^2}{2}}$$

Since $\left(1 + \frac{1}{n^2}\right)^{n^2} \to e$ $(n \to \infty)$ there is a large enough $n \in \mathbb{N}$ for which $\left(1 + \frac{1}{n^2}\right)^{n^2} \leq 2e$ and thus

$$|b_n| \leq \sqrt{2e}\frac{1}{2^n}.$$

From the convergence of the geometric series $\sum\limits_{n=0}^{\infty} \frac{1}{2^n}$, by the Majorant criterion, follows the absolute convergence of the given series.

(3) Let $c_n := \frac{i^n}{n}$ This series is not absolutely convergent because $\sum\limits_{n=1}^{\infty} |c_n| = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ and this diverges. However, the series is convergent: Due to

$$i^{4n-3} = i, \quad i^{4n-2} = -1, \quad i^{4n-1} = -i, \quad i^{4n} = 1 \quad \text{for } n \in \mathbb{N},$$

for all $N \in \mathbb{N}$ it follows

$$\begin{aligned} \sum_{k=1}^{4N} \frac{i^k}{k} &= \sum_{n=1}^{N} \Big(\frac{i}{4n-3} + \frac{-1}{4n-2} + \frac{-i}{4n-1} + \frac{1}{4n}\Big) \\ &= \sum_{n=1}^{N} \Big(\frac{-2}{4n(4n-2)} + i\frac{2}{(4n-1)(4n-3)}\Big), \end{aligned}$$

which is convergent for $N \to \infty$. Moreover, since for $j \in \{1, 2, 3\}$ we have

$$\Big|\sum_{k=1}^{4N+j} \frac{i^k}{k} - \sum_{k=1}^{4N} \frac{i^k}{k}\Big| \leq \sum_{k=4N+1}^{4N+j} \frac{1}{k} \leq \frac{3}{4N+1} \to 0 \quad \text{as } N \to \infty$$

we can conclude that the given series converges.

(b) For $z = x + iy$ we have

$$\begin{aligned} \sin(z) &= \frac{e^{ix-y} - e^{-ix+y}}{2i} = \frac{\big(\cos(x) + i\sin(x)\big)e^{-y} + \big(\cos(-x) + i\sin(-x)\big)e^{y}}{2i} \\ &= \frac{\sin(x)(e^y + e^{-y})}{2} + i\frac{\cos(x)(e^y - e^{-y})}{2} = \sin(x)\cosh(y) + i\cos(x)\sinh(y) \end{aligned}$$

and therefore

$$\begin{aligned} |\sin(z)|^2 &= \sin^2(x)\cosh^2(y) + \cos^2(x)\sinh^2(y) \\ &= \sin^2(x)\cosh^2(y) - \sin^2(x)\sinh^2(y) + \sin^2(x)\sinh^2(y) + \cos^2(x)\sinh^2(y) = \sin^2(x) + \sinh^2(y) \end{aligned}$$

The function $z \mapsto \sin(z)$ is unbounded in $\mathbb{C}$ since for $y \in \mathbb{R}$ we have

$$|\sin(iy)| = |\sinh(y)| \to \infty \quad \text{as } |y| \to \infty.$$

Exercise 2

(a) Let $a_n := (1+ni)^n$. Then it holds

$$\sqrt[n]{|a_n|} = |1+ni| = \sqrt{1+n^2} \to \infty \quad \text{as } n \to \infty,$$

so the radius of convergence is

$$R = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}} = 0.$$

So the power series converges only in $z = 0$.

(b) Let $b_n = in^2 + 2^n$. Then, for all $n \in \mathbb{N}$, it holds

$$\sqrt[n]{|b_n|} = \sqrt[n]{|in^2+2^n|} \geq \sqrt[n]{|2^n|} = 2$$

In addition, there exists $n_0 \in \mathbb{N}$ with $n^2 \leq 2^n$ for all $n \geq n_0$ and thus

$$\sqrt[n]{|b_n|} = \sqrt[n]{|in^2+2^n|} \leq \sqrt[n]{2^n+2^n} \to 2 \quad \leq \sqrt[n]{2}2 \quad \text{as } n \to \infty.$$

Therefore, it follows that $\sqrt[n]{|b_n|} \to 2$ as $n \to \infty$ and so the power series $\sum\limits_{n=1}^{\infty} b_n w^n$ has the convergence radius $R = \frac{1}{2}$. For $w = z^2$, we get that the given power series has the convergence radius $R = \frac{\sqrt{2}}{2}$.

(c) Let $c_n := (1+i^n)^{(n+1)/2}$. Since $|1+i^n| \leq |1|+|i^n| = 2$ for all $n \in \mathbb{N}_0$ and $|1+i^n| = 2$ for all $n = 4k$ ($k \in \mathbb{N}_0$) it foloows

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = \lim_{n\to\infty} \sqrt[n]{2^{(n+1)/2}} = \lim_{n\to\infty} 2^{(1+1/n)/2} = 2^{1/2}.$$

Therefore, the convergence radius of the given power series is $R = \frac{\sqrt{2}}{2}$.

(d) Let $d_n := \sum\limits_{k=1}^{n} \frac{1}{k+i}$. For all $k \in \mathbb{N}$ we have $|k+i| \geq Im(k+i) = 1$ and so

$$|d_n| \leq \sum_{k=1}^{n} \frac{1}{|k+i|} \leq \sum_{k=1}^{n} 1 = n.$$

On the other hand, it holds

$$|d_n| \geq Red_n = \sum_{k=1}^{n} \frac{k}{k^2+1} \geq \frac{1}{2}$$

Since both $\sqrt[n]{n} \to 1$ and $\sqrt[n]{\frac{1}{2}} \to 1$ as $n \to \infty$ hold, we conclude that the radius of convergence for the given series is $R = 1$.

Exercise 3 For z with $|z| < 1$ define

$$g(z) := \frac{\exp\left(\sum\limits_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}\right)}{1+z}$$

Using that $\sum\limits_{k=1}^{\infty} (-1)^k z^k = \frac{1}{1+z}$, we can conclude that

$$g'(z) = 0$$

and thus

$$\exp\left(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}\right) = C(1+z), \quad C = const.$$

For $z = 0$ we have

$$\exp(0) = C \Rightarrow C = 1.$$

Exercise 4

(a) We are looking for $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ such that $S(z) := \frac{az+b}{cz+d}$ satisfies the given conditions.

$$\begin{aligned} S(0) = -2 \quad &\Rightarrow \quad b = -2d, \\ S(2) = 0 \quad &\Rightarrow \quad a = -\frac{1}{2}b = d, \\ S(i) = \infty \quad &\Rightarrow \quad c = di, \\ S(\infty) = -i \quad &\Rightarrow \quad c = di. \end{aligned}$$

Since $ad - bc = d^2 + 2id^2 \neq 0$, it follows that

$$S(z) = \frac{dz - 2d}{diz + di} = \frac{z-2}{iz-1}.$$

(b) We are looking for $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{C}$ with $\tilde{a}\tilde{d} - \tilde{b}\tilde{c} \neq 0$ such that $T(z) := \frac{\tilde{a}z+\tilde{b}}{\tilde{c}z+\tilde{d}}$ satisfies the given conditions.

$$\begin{aligned} T(3i) = 0 \quad &\Rightarrow \quad \tilde{b} = -3\tilde{a}i, \\ T(0) = \infty \quad &\Rightarrow \quad \tilde{d} = 0, \\ T(i) = 1 \quad &\Rightarrow \quad \tilde{c} = -2\tilde{a} \end{aligned}$$

but $T(1) = \frac{\tilde{a}-3\tilde{a}i}{-2\tilde{a}} \neq i$. Thus there exists no Möbius tranformation which satisfies the given conditions.

(c) To determine S^{-1}, we have to find $p, q, r, s \in \mathbb{C}$ with $ps - qr \neq 0$ such that $S^{-1}(z) := \frac{pz+q}{rz+s}$ satisfies

$$\begin{aligned} S^{-1}(-2) &= 0, \\ S^{-1}(0) &= 2, \\ S^{-1}(\infty) &= 0, \\ S^{-1}(-i) &= \infty. \end{aligned}$$

Analogously to (a), we find that $p = 1, q = 2, r = -i, s = 1$, i.e.

$$S^{-1}(z) = \frac{z+2}{-iz+1}.$$

In order to find the fixed points of S we have to solve

$$S(z) = z \Longleftrightarrow z^2 = 2i \Longleftrightarrow z_{1,2} = \pm(1+i)$$