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# Funktionentheorie
# Exercise Sheet 6
# Solutions

**Exercise 1** The ring $G$ can be written as $G = G_1 \cap G_2$ with

$$G_1 = \{z \in \mathbb{C} : |z| > 1\},$$
$$G_2 = \{z \in \mathbb{C} : |z| < 2\},$$

First we determine $S(G_1)$. The boundary of $G_1$ is the unit circle. We know that the image of a circle under a Möbius transformation is a generalized circle. In order to find $S(\partial G_1)$ it suffices to find the image of 3 points. Since

$$S(-1) = -\frac{1}{2}, \quad S(i) = \frac{i-1}{2}, \quad S(-i) = \frac{-1-i}{2},$$

it follows that $S(\partial G_1) = \{w \in \mathbb{C} : Re\, w = -\frac{1}{2}\}$. Since $S$ is continuous and $G_1$ is connected, $S(G_1)$ is also connected. From the injectivity of $S$ we conclude that $S(G_1)$ and $S(\partial G_1)$ are disjoint. Moreover, we have

$$S(G_1) \subset \{\, w \in \mathbb{C} : Re\, w < -\frac{1}{2} \,\} \text{ because } \infty \in G_1 \text{ and } S(\infty) = -1,$$
$$S(\mathbb{D}) \subset \{\, w \in \mathbb{C} : Re\, w > -\frac{1}{2} \,\} \text{ (} \mathbb{D} \text{ is the open unit disc) because } 0 \in \mathbb{D} \text{ and } S(0) = 0.$$

From the surjectivity of $S$ we conclude that

$$S(G_1) = \{\, w \in \mathbb{C} : Re\, w < -\frac{1}{2} \,\}$$

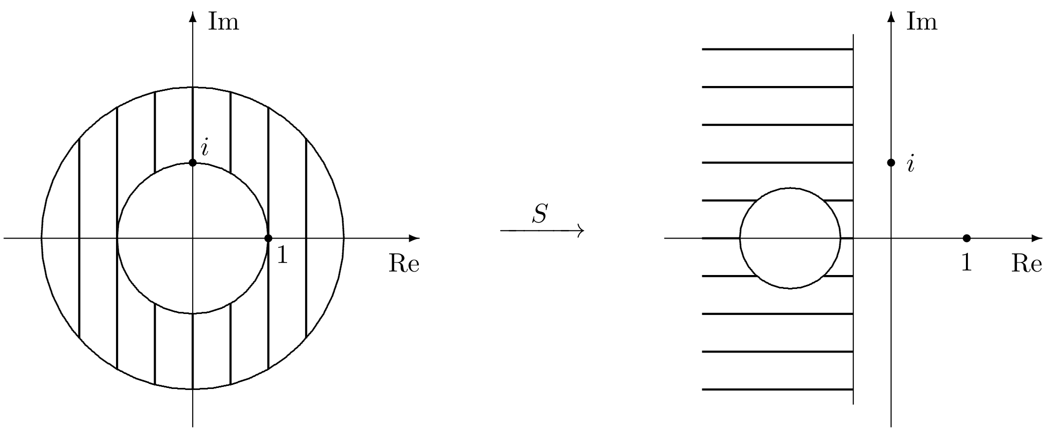
Now we determine $S(G_2)$. Since $S(2) = -2$, $S(-2) = -\frac{2}{3}$ and $\infty \notin S(\partial G_2)$, it follows that $S(\partial G_2)$ is a circle through the points $-2$ and $-\frac{2}{3}$. Due to the special form of the Möbius transformation $S$ we have that $\overline{S(z)} = S(\overline{z})$. Moreover, we note that $\partial G_2$ is symmetric with respect to reflections at the real axis. Therefore, it holds

$$\begin{aligned} w \in S(\partial G_2) &\iff w = S(z) \text{ and } z \in \partial G_2 \iff \overline{w} = \overline{S(z)} \text{ and } \overline{z} \in \partial G_2 \\ &\iff \overline{w} = S(\overline{z}) \text{ and } \overline{z} \in \partial G_2 \iff \overline{w} \in S(\partial G_2)\,, \end{aligned}$$

i.e. $S(\partial G_2)$ is symmetric with respect to reflections at the real axis. Thus $S(\partial G_2)$ is the circle with center $-\frac{4}{3}$ and radius $\frac{2}{3}$ and $S(G_2)$ is either the interior or the exterior of this circle. The same argumentation as for $S(G_1)$ show that $S(G_2)$ is the exterior of the circle.

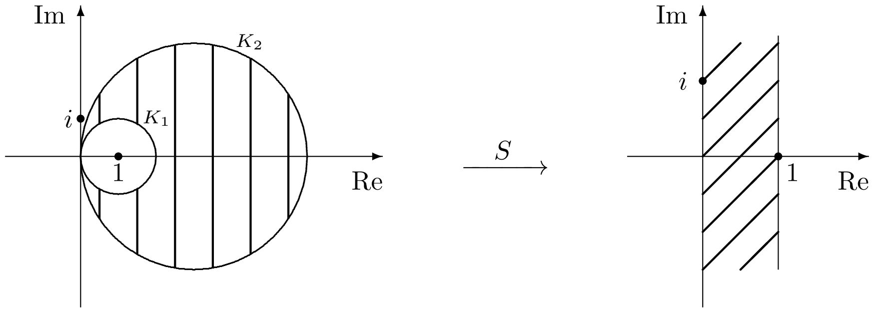
Since $S$ is injective, it follows

$$S(G) = S(G_1) \cap S(G_2) = \left\{\, w \in \mathbb{C} : Re\, w < -\frac{1}{2}\,,\ |w + \tfrac{4}{3}| > \tfrac{2}{3} \,\right\}.$$



**Exercise 2**

The Möbius transformation $S$ should do




Since $K_1$ and $K_2$ intersect in 0 it follows that we must have $S(0) = \infty$, i.e., $S$ must be of the following form

$$S(z) = \frac{az+b}{cz}\,, \quad S(z) = \alpha + \frac{\beta}{z}\,.$$

Let us see where $G$ is mapped by $T(z) := 1/z$. From $T(0) = \infty$ we see that $T(K_1)$ and $T(K_2)$ are lines. Since $K_1$ and $K_2$ are symmetric with respect to reflections at the real axis and since $\overline{T(z)} = T(\overline{z})$ holds, analogously to the previous exercese, we conclude that $T(K_1)$ and $T(K_2)$ are symmetric with respect to reflections at the real axis. Therefore $T(K_1)$ and $T(K_2)$ are lines perpendicular to the real axis. From $2 \in K_1$, $T(2) = \frac{1}{2}$ and $6 \in K_2$, $T(6) = \frac{1}{6}$ it follows that $T(G) = \{\, w \in \mathbb{C} : \frac{1}{6} < \mathrm{Re} w < \frac{1}{2}\,\}$, i.e., $T(G)$ is a stripe of width $\frac{1}{3}$. By $z \mapsto 3/z$ $G$ will be mapped to the stripe $\{\, w \in \mathbb{C} : \frac{1}{2} < Rew < \frac{3}{2}\,\}$ which has width 1. We only have to shift it now. Therefore, by choosing $\beta = 3$ and $\alpha = -\frac{1}{2}$ we obtain the desired transformation.

$$S(z) = -\frac{1}{2} + \frac{3}{z} = \frac{-z+6}{2z}\,.$$

This transformation is not uniquely determined since we could have chosen e.g. $\beta = 3$ and $\alpha = -\frac{1}{2} + i$.

**Exercise 3** We note that the reflection point of $a$ with respect to the imaginary axis is $-\bar{a}$. Furthermore, from the reflection principle, it follows that $S(a) = 0$ is the reflection point of $S(-\bar{a}) = \infty$ with respect to $S(\bar{\mathbb{I}})$. Therefore, $S(\bar{\mathbb{I}})$ must be a circle centered at 0. Due to

$$|S(0)| = |e^{it}|\left|\frac{0-a}{0+\bar{a}}\right| = 1$$

it follows that the radius of this circle is 1. Thus, the left half-plane will be mapped either to the interior or the exterior of the unit circle. Since $S(a) = 0$, it's mapped to the interior.
Now the other way around - assume that $S$ is a Möbius transformation that maps the left half-plane to the unit ball. Then the point $a = S^{-1}(0)$ lies in the left half-plane, i.e., we have $Re\ a < 0$. Since the reflection point of $a$ with respect to the imaginary axis is $-\bar{a}$ and $S(a) = 0$, from the reflection principle, it follows that $S(-\bar{a}) = \infty$. Therefore, $S$ has the form

$$S(z) = k\frac{z-a}{z+\bar{a}} \quad (k \in \mathbb{C}, k \neq 0).$$

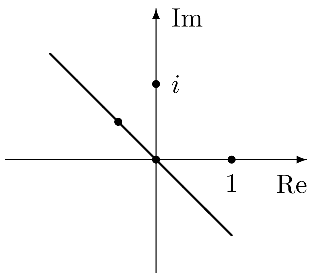
But since $S(0)$ has to lie at the unit circle and $|S(0)| = |k|$, it follows that $|k| = 1$. Therefore, $k = e^{it} (t \in \mathbb{R})$.

**Exercise 4**

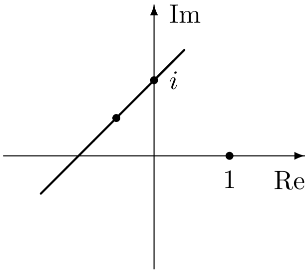
(a) Since $S$ maps generalized circles into generalized circles, the images under $S$ of the unit circle as well as of the axes will be generalized circles. It holds

$$S(1) = \tfrac{1}{2}(i-1)\,, \qquad S(i) = 0\,, \qquad S(-1) = \infty\,, \qquad S(0) = i\,, \qquad S(\infty) = -1.$$

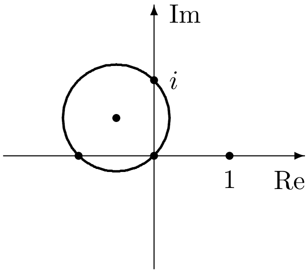
For $\partial\mathbb{E}$: 1, $i$, $-1 \in \partial\mathbb{E}$ are mapped to $\frac{1}{2}(i-1)$, 0 and $\infty$, resp. Therefore, the image of the unit circle is $\{z \in \mathbb{C} : \mathrm{Im} z = -\mathrm{Re} z\}$:



For the real axis: 1, $-1$ and 0 are mapped to $\frac{1}{2}(i-1)$, $\infty$ and $i$, resp. Therefore, the image of the real axis is $\{z \in \mathbb{C} : \mathrm{Im} z = 1 + \mathrm{Re} z\}$:




For imainary axis: $i$, 0 and $\infty$ are mapped to 0, $i$ and $-1$, resp. Therefore, the image of the imaginary axis is the circle with center $\frac{1}{2}(i-1)$ and radius $\frac{1}{2}\sqrt{2}$.




(b) Since $S(\partial\mathbb{E})$ is a line through 0, for the Möbius transformation $S_1(z) := 2z$ it holds $S_1(S(\partial\mathbb{E})) = S(\partial\mathbb{E})$. Therefore, the transformation we are looking for is $T := S_1 \circ S$, i.e.

$$T(z) = 2S(z) = \frac{2i - 2z}{1 + z}.$$

Another possibility would be $T(z) := S(-z)$, i.e.,

$$T(z) = \frac{i + z}{1 - z},$$

because for the Möbius transformation $S_2(z) := -z$ it holds: $S_2(\partial\mathbb{E}) = \partial\mathbb{E}$.