Karlsruher Institut für Technologie
Institut für Analysis
Dr. A. Müller-Rettkowski
M. Sc. S. Avramska

SS 2012
12.06.2012

# Funktionentheorie
# Exercise Sheet 8
# Solutions

### Exercise 1

(a) In the star-shaped domain $\mathbb{C}$ the function $f(z) := \cos(\pi z)$ is holomorph. Since $a = 0$ lies in the interior of the integration path, the Cauchy integral formula gives us

$$\int\limits_{|z|=1} \frac{f(z)}{z-0} dz = 2\pi i f(0) = 2\pi i.$$

(b) We set $f(z) = \cos(\pi z^2) + \sin(\pi z^2)$. As above, the Cauchy integral formula gives

$$\int\limits_{|z|=3} \frac{f(z)}{(z-1)(z-2)} dz = \int\limits_{|z|=3} \frac{f(z)}{z-2} dz - \int\limits_{|z|=3} \frac{f(z)}{z-1} dz = 2\pi i f(2) - 2\pi i f(1) = 4\pi i.$$

(c) The integrand can be written as

$$\frac{e^z}{z^2+2z} = \frac{1}{2}\Big(\frac{e^z}{z} - \frac{e^z}{z+2}\Big).$$

As above, from the Cauchy integral formula it follows

$$\int\limits_{|z|=3} \frac{e^z}{z^2+2z} dz = \frac{1}{2}\Big( \int\limits_{|z|=3} \frac{e^z}{z} dz - \int\limits_{|z|=3} \frac{e^z}{z+2} dz\Big) = (1-e^{-2})\pi i.$$

(d) Obviously, the integrand is holomorph in the disc $\{z \in \mathbb{C} : |z| \leq 1\}$. Therefore, from the Cauchy integral theorem, it follows that the value of the integral is 0.

### Exercise 2

(a) In the star-shaped domain $\mathbb{C}$ the function $f(z) := e^{-z^2}$ is holomorph. Furthermore, $\gamma = \gamma_1 * \gamma_2 * \gamma_3^-$ is a closed curve. Using the notation $I(\Gamma) = \int\limits_\Gamma f(z)dz$, from the Cauchy integral formula, we get

$$0 = I(\gamma) = I(\gamma_1) + I(\gamma_2) + I(\gamma_3^-) = I(\gamma_1) + I(\gamma_2) - I(\gamma_3).$$

(b) It holds $z_2^2(t) = (R+it)^2 = R^2 + 2iRt - t^2$ and $z_2'(t) = i$. Therefore, we get

$$\Big|\int\limits_{\gamma_2} e^{-z^2} dz\Big| \leq \int\limits_0^R |e^{-z_2^2(t)} z_2'(t)| dt = \int\limits_0^R e^{t^2-R^2} dt.$$

Since for all $t \in [0,R]$ we have $t^2 \leq Rt$, we can conclude that

$$\Big|\int\limits_{\gamma_2} e^{-z^2} dz\Big| \leq \int\limits_0^R e^{Rt-R^2} dt = \Big[\frac{e^{Rt}-e^{R^2}}{R}\Big]_{t=0}^R = \frac{1-e^{-R^2}}{R} \xrightarrow{R\to\infty} 0 \quad \text{as } R \to \infty.$$

(c) Now we consider $I(\gamma_1)$ and $I(\gamma_3)$. We have

$$I(\gamma_1) = \int\limits_0^R e^{-z_1^2(t)} z_1'(t) dt = \int\limits_0^R e^{-t^2} dt \xrightarrow{R\to\infty} \int\limits_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}.$$

Furthermore, since $z_3^2(t) = t^2(1+i)^2 = 2it^2$ and $z_3'(t) = 1+i$, it follows

$$I(\gamma_3) = \int_0^R e^{-z_3^2(t)} z_3'(t)dt = \int_0^R e^{-2it^2}(1+i)dt \xrightarrow{R\to\infty} (1+i)\int_0^\infty e^{-2it^2}dt$$

Using the substitution $x = \sqrt{2}t$, we get

$$I(\gamma_3) \xrightarrow{R\to\infty} (1+i)\int_0^\infty e^{-ix^2}\frac{dx}{\sqrt{2}} = \frac{1+i}{\sqrt{2}}\int_0^\infty [\cos(x^2) - i\sin(x^2)]dx$$

Therefore, using (a), we can conclude that

$$\frac{1+i}{\sqrt{2}}\int_0^\infty [\cos(x^2) - i\sin(x^2)]dx = \frac{\sqrt{\pi}}{2}. \tag{1}$$

Setting $C := \int_0^\infty \cos(x^2)dx$ and $S := \int_0^\infty \sin(x^2)dx$, we can rewrite (1) as follows

$$(1+i)(C - iS) = \frac{\sqrt{2\pi}}{2}, \quad \text{i.e. } (C+S) + i(C-S) = \frac{\sqrt{2\pi}}{2}.$$

Hence $C = S = \frac{\sqrt{2\pi}}{4}$.

## Exercise 3

(a) $\gamma_R$ can be parametrized as $\gamma_R(t) = Re^{it}$ with $0 \le t \le 2\pi$. For $p$ we use its power series representation $p(z) = \sum_{k=0}^n a_k z^k$. Then we have

$$\int_{\gamma_R} \overline{p(z)}dz = \int_0^{2\pi} \overline{p(\gamma_R(t))}\gamma_R'(t)dt = \int_0^{2\pi} \sum_{k=0}^n \overline{a_k R^k e^{ikt}} iRe^{it}dt$$

$$= iR\int_0^{2\pi} \sum_{k=0}^n \overline{a_k} R^k e^{-ikt}e^{it}dt = iR\sum_{k=0}^n \overline{a_k}R^k \int_0^{2\pi} e^{i(1-k)t}dt.$$

Using the notation $I_k := \int_0^{2\pi} e^{i(1-k)t}dt$, we get

$$I_1 = \int_0^{2\pi} 1dt = 2\pi \quad \text{and} \quad I_k = \left[\frac{e^{i(1-k)t}}{i(1-k)}\right]\Big|_{t=0}^{2\pi} = \frac{1-1}{i(1-k)} = 0 \text{ for } k \neq 1.$$

On the other hand, we notice that $p'(0) = \sum_{k=1}^n k a_k z^{k-1}\big|_{z=0} = a_1$. Therefore, we have

$$\int_{\gamma_R} \overline{p(z)}dz = iR\sum_{k=0}^n \overline{a_k}R^k I_k = iR2\pi\overline{z_1}R = 2\pi i R^2\overline{p'(0)}.$$

(b) The curve $\gamma$ goes from $\gamma(0) = 1$ to $\gamma(\frac{\pi}{2}) = i$. Since in $\mathbb{C}$ the primitive function of $\cos(z)$ is $\sin(z)$, we have

$$\int_\gamma \cos(z)dz = \sin(i) - \sin(0) = \frac{e^{i\cdot i} - e^{-i\cdot i}}{2i} = \frac{i(e^2-1)}{2e}.$$

For the lenght of $\gamma$:

$$\gamma'(t) = ie^{it}\sin(t) + e^{it}\cos(t) = e^{2it}.$$

Hence,

$$l(\gamma) = \int_0^{\pi/2} |\gamma'(t)|dt = \int_0^{\pi/2} 1dt = \frac{\pi}{2}.$$

(c) We denote the two real integrals by $I_1$ and $I_2$. Using the parametrization of the unit circle $\gamma(t) := e^{it}$ with $0 \le t \le 2\pi$, we get

$$I_1 + iI_2 = \int_0^{2\pi} e^{-\sin t}\big[\cos(t+\cos t) + i\sin(t+\cos t)\big]\,dt = \int_0^{2\pi} e^{i\sin t} e^{i(t+\cos t)}\,dt$$
$$= \int_0^{2\pi} e^{i(\cos t + i\sin t)} e^{it}\,dt = \int_0^{2\pi} \frac{e^{ie^{it}}}{i} ie^{it}\,dt = \int_0^{2\pi} \frac{e^{i\gamma(t)}}{i}\gamma'(t)\,dt = \int_\gamma \frac{e^{iz}}{i}\,dz.$$

In the whole of $\mathbb{C}$ the function $z \mapsto e^{iz}/i$ has the primitive function $z \mapsto -e^{iz}$. Since $\gamma$ is a closed path in $\mathbb{C}$, it follows that

$$I_1 + iI_2 = \int_\gamma \frac{e^{iz}}{i}\,dz = 0.$$

Hence $I_1 = 0$ and $I_2 = 0$.