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Funktionentheorie
Exercise Sheet 9
Solutions

Exercise 1

(a) Using that

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$$

and the Cauchy product formula, we get

$$g(z) = \frac{f(z)}{1-z} = \Big(\sum_{k=0}^{\infty} a_k z^k\Big)\Big(\sum_{k=0}^{\infty} z^k\Big) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k z^k z^{n-k} = \sum_{n=0}^{\infty}\big(\sum_{k=0}^{n} a_k\big) z^n$$

By induction, it can be shown that

$$g^{(n)}(0) = n!\sum_{k=0}^{n} a_k \quad \text{for all } n \in \mathbb{N}_0$$

(b) First we note that for $0 < r < 1$ the fucntion g is holomorph. Then, using the Cauchy integral formula, we get

$$\frac{1}{2\pi i}\int\limits_{|z|=r} \frac{f(z)}{(1-z)z^{n+1}}dz = \frac{1}{2\pi i}\int\limits_{|z|=r} \frac{g(z)}{z^{n+1}}dz = \frac{g^{(n)}(0)}{n!} \overset{(a)}{=} \sum_{k=0}^{n} a_k$$

(c) By (b), for $f(z) = e^z = \sum\limits_{k=0}^{\infty} \frac{z^k}{k!}$ and $n = 2$ we have

$$\int\limits_{|z|=r} \frac{e^z}{(1-z)z^3}dz = 2\pi i \sum_{k=0}^{2} \frac{1}{k!} = 5\pi i.$$

For the second integral we have to consider $f(z) = \frac{e^z}{1-z}$ and $n = 2$.
Since for $|z| < 1$

$$f(z) = (1 + z + \frac{1}{2}z^2 + \cdots)(1 + z + z^2 + \cdots) = 1 + 2z + \frac{5}{2}z^2 + \cdots$$

it follows that

$$\int\limits_{|z|=r} \frac{e^z}{(1-z)^2 z^3}dz = 2\pi i(1 + 2 + \frac{5}{2}) = 11\pi i.$$

Exercise 2

(a) Let M denote the interior of the closed curve C. Assume that f has infinitely many c-Stellen in M. Let us call them $z_1, z_2, \dots$. Since $\bar{M}$ is bounded and closed, it follows that there exists a subsequence $\{z_{k_l}\}_{k_l \in \mathbb{N}}$ of $\{z_k\}_{k\in\mathbb{N}}$ such that $z_{k_l} \to z_0$ with $z_0 \in \bar{M} \subset G$. Therefore, by the Identity theorem, it follows that $f = c$ in G, which is a contradiction.

(b) $\sin\left(\frac{1}{1-z}\right) = 0 \iff \frac{1}{1-z} = k\pi \;\; (k \in \mathbb{Z}) \iff z = 1 - \frac{1}{k\pi} \;\; (k \in \mathbb{Z})$

Note: Although there are infinitely many zeroes of $\sin\left(\frac{1}{1-z}\right)$, there is no contradiction to (a), since in this case the accumulation point of the zeroes lies on the boundary of $G = \{z \in \mathbb{C} : |z| < 1\}$.

Exercise 3 If the funtion f is a polynomial of degree $\leq n$, then

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n \,,$$

If we set $b := |a_0| + |a_1| + \cdots + |a_n|$, then for $|z| \geq 1$ the following estimate holds

$$|f(z)| \leq |a_0| + |a_1 z| + \cdots + |a_n z^n| \leq |a_0| \cdot |z|^n + \cdots + |a_n| \cdot |z|^n = b|z|^n \,.$$

Since f is countinuous, it follows that $a := \max_{|z|\leq 1} |f(z)| < \infty$. Thus we have

$$|f(z)| \leq a \quad \text{for } |z| \leq 1 \qquad \text{and} \qquad |f(z)| \leq b|z|^n \quad \text{for } |z| \geq 1 \,,$$

i.e., $|f(z)| \leq a + b|z|^n$ for all $z \in \mathbb{C}$.

Now the other direction: we assume that the estimate holds. The Cauchy integral formula for derivatives gives us for all $m \in \mathbb{N}_0$ and $r > 0$

$$f^{(m)}(0) = \frac{m!}{2\pi i} \int\limits_{|z|=r} \frac{f(z)}{(z-0)^{m+1}} \, dz \,.$$

For $|z| = r$ the following estimate holds

$$\left| \frac{f(z)}{(z-0)^{m+1}} \right| \leq \frac{a + b|z|^n}{|z|^{m+1}} = \frac{a + br^n}{r^{m+1}} \,.$$

Since the circle $|z| = r$ has lenght $2\pi r$ it follows that

$$|f^{(m)}(0)| \leq \frac{m!}{2\pi} \cdot 2\pi r \cdot \frac{a + br^n}{r^{m+1}} = \frac{ar + br^{n+1}}{r^{m+1}/m!} \xrightarrow{r\to\infty} 0 \qquad \text{for } m > n \,.$$

This means that $f^{(m)}(0) = 0$ for $m > n$, i.e. the power series representation of f at the point 0 contains only powers $\leq n$. In other words, f is a polynomial of degree $\leq n$.

Exercise 4

(a) Let $D(z_0, \delta) \subset G$. Since f is continuous, we have that either $f = \mathit{const}$ or, by Theorem 4 (Chapter 10.4), $f\big(D(z_0,\delta)\big)$ is a domain. Since we have $|f(z)| \leq |f(z_0)|, z \in D(z_0, \delta)$, it follows that $f\big(D(z_0,\delta)\big)$ cannot contain an open disc around $f(z_0)$. Thus $f \equiv \mathit{const}$.

(b) Assume that $f(z_0) \neq 0$ and consider $g := \frac{1}{f}$. If $z_0 \in G$ is a local Min of $|f|$, then $z_0 \in G$ is a local Max of $|g|$ and thus, by the Maximum Princip, it follows that $g = \mathit{const}$ and therefore $f = \mathit{const}$.