Functional Analysis

7th Exercise Sheet

Exercise 23:
Let $X$ be a Banach space and $Y$ be separable Banach space with the following property: There is a sequence of finite-rank operators $S_n \subseteq F(Y)$ such that
\[ \lim_{n \to \infty} S_n y = y, \quad \forall y \in Y. \]
Then the following holds $F(X, Y) = K(X, Y)$.

Hint: Show $\|S_n T - T\| \to 0$. Start by covering $T(B_X)$ with finitely many balls of radius $\epsilon > 0$.

Exercise 24:
Let $1 \leq p < \infty$ and $A \subseteq l^p$. Then the following are equivalent:

1. $A$ is relatively compact
2. $A$ is bounded and $\lim_{n \to \infty} \sup_{x \in A} \left( \sum_{j=n}^{\infty} |x_j|^p \right)^{\frac{1}{p}} = 0$

Exercise 25:
Consider $C^\alpha([0,1])$ with $0 < \alpha < 1$ and norm $\|f\|_{C^\alpha} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^{\alpha}}$. Show that the identity $I : (C^\alpha([0,1]), \|\cdot\|_{C^\alpha}) \to (C([0,1]), \|\cdot\|_\infty)$ is compact.

Exercise 26:
Let $\omega := K^\mathbb{N}$ be the space of all sequences $x = (x_n)_{n \in \mathbb{N}}$ in $K$. Show that $\omega$ is a complete metric space for
\[ d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|x_j - y_j|}{1 + |x_j - y_j|} \]
and
\[ \lim_{n \to \infty} d(x^{(n)}, x^{(0)}) = 0 \iff \forall j \in \mathbb{N} : \lim_{n \to \infty} x^{(n)}_j = x^{(0)}_j. \]

Exercise 27:
Let $X$ be a Banach space with $\dim X = \infty$. Then no basis of $X$ is countable.

Remark: The exercises marked with a star can be handed in for the correction till Wednesday noon to the letterbox at the ground level of the Math building 20.30.

http://www.math.kit.edu/iana1/edu/funcana2018w/en