Harmonic Analysis for Dispersive Equations

01. Problem Sheet

Exercise 1: Consider the function \( f(x) = e^{-x} \chi_{[0,\infty)}(x) \).

- Show that \( f \in L^1(\mathbb{R}) \), compute \( \hat{f} \) and show that \( \hat{f} \notin L^1(\mathbb{R}) \).
- Find the derivative, \( f' \), of \( f \) and show that the equality \( 2\pi i \xi \hat{f}(\xi) = \hat{f}'(\xi) \) does not hold. Where is the problem?

Exercise 2: Consider \( f \in L^1(\mathbb{R}) \), \( a \in \mathbb{R} \) and define the function \( g(x) = f(x + a) - f(x) \). Show that \( \hat{g} \) has infinitely many zeros.

Exercise 3: Translation in \( L^p(\mathbb{R}^d) \), \( 1 \leq p < \infty \): Consider \( f \in L^p(\mathbb{R}^d) \) and show that the map

\[ \mathbb{R}^d \ni x \rightarrow f_x \in L^p(\mathbb{R}^d), \]

where \( f_x(y) = f(y - x) \), is uniformly continuous.