Harmonic Analysis for Dispersive Equations

13. Problem Sheet

Exercise 37:
Consider the linear Schrödinger equation

\[ i \partial_t u = -\Delta u, \quad u(0, x) = u_0(x), \]  

where \( u \) denotes a complex valued function of spacetime \( \mathbb{R}_t \times \mathbb{R}^d_x \), and we assume the initial data \( u_0 \) to be in the Schwartz class. By taking Fourier transforms, we observe that

\[ \hat{u}(t, \xi) = e^{-4\pi^2 t|\xi|^2} \hat{u}_0(\xi), \]

which means that solutions with Schwartz initial data are also Schwartz for all times \( t \in \mathbb{R} \).

Denote by \( e^{it\Delta} \) the multiplier operator with symbol \( e^{-\frac{1}{4\sigma^2} + 2\pi i \sigma t} \). Prove the following:

- \( \| e^{it\Delta} u_0 \|_2 = \| u_0 \|_2 \) and \( \| \nabla e^{it\Delta} u_0 \|_2 = \| \nabla u_0 \|_2 \) for any \( u_0 \in L^2(\mathbb{R}^d) \). (Hint: Use Plancherel)

- Let \( u_0(x) = e^{-\frac{|x|^2}{4\sigma^2} + 2\pi i \sigma t} \), with \( \sigma > 0 \) and \( \xi_0 \in \mathbb{R}^d \). Show that the solution to (1) is given by the formula

\[ e^{it\Delta} u_0(x) = \left( \frac{\sigma^2}{\sigma^2 + it} \right)^\frac{d}{2} e^{-4\pi^2 t|\xi_0|^2 + 2\pi i \sigma t} e^{-\frac{|x - y|^2}{4\sigma^2} - \frac{|x - \xi_0|^2}{4(\sigma^2 + it)}}. \]  

(Hint: Use Fourier inversion to express the function \( e^{it\Delta} u_0(0) \) as an integral in \( \mathbb{R}^d \). After some calculations you will have to calculate an integral of the form \( \int_{\mathbb{R}} e^{-t^2} dt \), with \( \alpha, \beta \in \mathbb{C} \) and \( \text{Re}(\alpha^2) > 0 \). For this you have to use Cauchy’s integral theorem which states that if \( f \) is holomorphic in a domain \( \Omega \) then its path integral over a path \( \gamma \subset \Omega \) is 0 for every closed, piecewise continuously differentiable path \( \gamma \). This will allow you to pass to an integral of the form \( \int_{\mathbb{R}} e^{-t^2} dt \)

- Let \( u_0 \in S(\mathbb{R}^d) \). Show that the solution to (1) is given by the formula

\[ e^{it\Delta} u(x) = (4\pi i t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4it}} u_0(y) dy, \]

for \( t \neq 0 \), where the equality is meant in the \( L^2(\mathbb{R}^d) \) sense. (Hint: Start from the equality

\[ e^{it\Delta} \left[ (4\pi \sigma^2)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4\sigma^2}} u_0(y) dy \right] = [4\pi (\sigma^2 + it)]^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4(\sigma^2 + it)}} u_0(y) dy, \]

which follows from the fact that \( e^{it\Delta} \) passes inside the integral and the use of (2) with \( \xi_0 = 0 \). Then observe that from the properties of Dirac families.

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\[ \lim_{\sigma \to 0^+} (4\pi \sigma^2)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4\sigma^2}} u_0(y) dy = u_0(x), \]

both pointwise and in the \( L^2(\mathbb{R}^d) \) sense.

- Prove the dispersive estimates

\[ \|e^{it\Delta}u_0\|_p \lesssim |t|^{-\frac{d}{2}(1-\frac{2}{p})} \|u_0\|_{p'}, \]

for all \( 2 \leq p \leq \infty \). (Hint: Prove it first for \( p = \infty \) using the previous question and then interpolate using that \( e^{it\Delta} \) is an isometry in \( L^2(\mathbb{R}^d) \))

**Exercise 38:**
Show that there is a fixed Schwartz function \( \phi \) such that if \( f \in L^1(\mathbb{R}^d) + L^2(\mathbb{R}^d) \) and \( \hat{f} \) is supported in the ball \( B(0, R) \) then

\[ f = \phi^{(R^{-1})} \ast f, \]

where for a positive number \( \epsilon \) we define \( \phi^{(\epsilon)}(x) = e^{-d\phi(x/\epsilon)} \). (Hint: Take \( \phi \in S(\mathbb{R}^d) \) such that \( \hat{\phi} \) is equal to 1 on \( B(0, 1) \). Then calculate the Fourier transform of the function \( \phi^{(R^{-1})} \ast f - f \))

**Exercise 39:**
(Bernstein’s inequality for the ball) Suppose that \( f \in L^1(\mathbb{R}^d) + L^2(\mathbb{R}^d) \) and that \( \hat{f} \) is supported in the ball \( B(0, R) \). Then, for any \( 1 \leq p \leq q \leq \infty \)

\[ \|f\|_q \lesssim R^{(\frac{1}{p} - \frac{1}{q})} \|f\|_p. \]

(Hint: Consider the function \( \phi^{(R^{-1})} \) from the previous Exercise and write \( f = \phi^{(R^{-1})} \ast f \). Calculate \( \|\phi^{(R^{-1})}\|_r \) and apply Young’s inequality for convolutions for the indices \( 1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p} \))

**Exercise 40:**
For any dimension \( d \geq 1 \) and \( N \in \mathbb{R}_+ \) show that

\[ \|e^{\pm it|\nabla|}P_N f\|_{L^p_x} \lesssim (1 + N|t|)^{-\frac{d-1}{2}} N^d \|P_N f\|_{L^1_x}, \]

where \( e^{it|\nabla|} \) is the multiplier operator with symbol \( e^{2\pi it|\xi|} \), and \( P_N f \) are the Littlewood-Paley cutoff operators (denoted by \( f_j = \phi_j(D)f \) in class, \( j \in \mathbb{N} \)). Then, interpolating with the estimate \( \|e^{\pm it|\nabla|}P_N f\|_2 = \|P_N f\|_2 \), prove that for all \( 2 \leq p \leq \infty \) we have the estimate

\[ \|e^{\pm it|\nabla|}P_N f\|_{L^p_x} \lesssim (1 + N|t|)^{-\frac{d-1}{2}(1-\frac{2}{p})} N^d \|P_N f\|_{L^p_x}, \]

where \( p' \) is the conjugate exponent of \( p \). (Hint: For \( d = 1 \) or \( d \geq 2 \) and \( |t| \lessapprox N^{-1} \) the claim follows from Bernstein’s inequality for the ball. Therefore, it remains to deal with the case \( d \geq 2 \) and \( |t| \gg N^{-1} \). For this, consider the fattened Littlewood-Paley cutoff \( \tilde{P}_N = P_{N/2} + P_N + P_{2N} \) and write
\[ e^{it|\nabla|} P_N f = e^{it|\nabla|} \tilde{P}_N (P_N f) = \left[ e^{2\pi it|\xi|} \tilde{\psi} \left( \frac{\xi}{N} \right) \right] \ast P_N f, \]

where \( \tilde{\psi} \) is the Fourier multiplier associated with \( \tilde{P}_1 \). Hence, to derive the desired estimate it suffices to show

\[
\left| \int_{\mathbb{R}^d} e^{ix\xi + it|\xi|} \tilde{\psi} \left( \frac{\xi}{N} \right) d\xi \right| \lesssim N^{\frac{d+1}{2}} |t|^{-\frac{d+1}{2}}.
\]

Consider the following two cases: 1) \( |x| \ll |t| \), where the phase function has no critical points and integration by parts should be applied to estimate the oscillatory integral, and 2) \( |t| \lesssim |x| \) and the estimate \( |\tilde{\sigma}(x)| \lesssim \langle x \rangle^{-\frac{d+1}{2}} \) which was proven in class should be used in order to estimate the oscillatory integral.

http://www.math.kit.edu/iana1/lehre/harmaanadispeqn2017w/en