Harmonic Analysis for Dispersive Equations

03. Problem Sheet

**Exercise 9:** Suppose that \( f \) and \( \hat{f} \) are both in \( L^1(\mathbb{R}^d) \). Show that the inversion formula holds for a.e. \( x \in \mathbb{R}^d \) and that there is a continuous, bounded function \( g \in L^1(\mathbb{R}^d) \) such that \( f(x) = g(x) \) for a.e. \( x \in \mathbb{R}^d \).

**Exercise 10:** Show that the Fourier transform \( \mathcal{F} \) is a homeomorphism from \( S(\mathbb{R}^d) \) onto itself, that is 1-1, continuous, onto, and with a continuous inverse.

**Exercise 11:** For a fixed function \( g \in S(\mathbb{R}^d) \) (called the window function) the Short Time Fourier Transform (STFT) of a function \( f \) with respect to \( g \) is defined as

\[
V_g f(x, \omega) = \int_{\mathbb{R}^d} f(t) \bar{g}(t-x) e^{-2\pi i \omega \cdot t} dt,
\]

for \( x, \omega \in \mathbb{R}^d \).

- Show that if \( f \in L^2(\mathbb{R}^d) \) then \( V_g f \) is uniformly continuous on \( \mathbb{R}^{2d} \) and that the following identity holds

\[
V_g f(x, \omega) = \mathcal{F}(f \cdot T_x \bar{g})(\omega),
\]

where \( T_x \) is the translation operator \( T_x f(t) = f(t-x) \) and \( M_\omega \) the modulation operator \( M_\omega f(t) = e^{-2\pi i \omega \cdot t} f(t) \). Notice that the definition of the Short Time Fourier Transform makes sense if the window function \( g \) is in \( L^2(\mathbb{R}^d) \).

- Consider \( f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d) \). Then \( V_{g_1} f_j \in L^2(\mathbb{R}^{2d}) \) for \( j = 1, 2 \) and

\[
\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^d)} \overline{\langle g_1, g_2 \rangle}_{L^2(\mathbb{R}^d)}.
\]

This identity is called the orthogonality relations for STFT. Conclude that if \( f, g \in L^2(\mathbb{R}^d) \) then \( \|V_g f\|_2 = \|f\|_2 \|g\|_2 \). In particular, if \( \|g\|_2 = 1 \) then \( \|V_g f\|_2 = \|f\|_2 \). Therefore, in this case the STFT is an isometry from \( L^2(\mathbb{R}^d) \) to \( L^2(\mathbb{R}^{2d}) \).

- Fix \( g, \gamma \in L^2(\mathbb{R}^d) \) and let \( K_n \subset \mathbb{R}^{2d} \) be a nested exhausting sequence of compact sets for \( n \in \mathbb{N} \). Define \( f_n \) as

\[
f_n(t) = \frac{1}{\langle \gamma, g \rangle} \int K_n V_g f(x, \omega) M_\omega T_x \gamma(t) \, dx \, d\omega.
\]

Prove that \( \|f - f_n\|_2 \to 0 \) as \( n \to \infty \).
Exercise 12: Let \( f, g : \mathbb{R}^d \rightarrow \mathbb{C} \) be a measurable functions. The decreasing rearrangement of \( f \) is the function \( f^* : [0, \infty) \rightarrow [0, \infty] \) defined by

\[
f^*(t) = \inf\{s \in [0, \infty) : d_f(s) \leq t\},
\]

where we set \( \inf \emptyset = \infty \) and \( d_f \) is the distribution function of \( f \). Prove the following properties:

- The function \( f^* \) is decreasing, right continuous and for all \( s, t \geq 0 \)

\[f^*(t) > s \iff d_f(s) > t.\]

- If \( |f(x)| \leq |g(x)| \) for almost all \( x \), then \( f^* \leq g^* \).

- The functions \( f \) and \( f^* \) are equimeasurable, that is

\[
|\{x \in \mathbb{R}^d : |f(x)| > s\}| = |\{t \in [0, \infty) : f^*(t) > s\}|,
\]

and for all \( 1 \leq p < \infty \)

\[
\|f\|_{L^p(\mathbb{R}^d)} = \|f^*\|_{L^p(0, \infty)}.
\]

For \( p = \infty \) we have \( \|f\|_{\infty} = \sup f^* = f^*(0) \).

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