Exercise 13:
For \( g : (0, \infty) \to \mathbb{R} \) measurable define its Hardy-Littlewood operator on \((0, \infty)\) by

\[
Tg(x) = \frac{1}{x} \int_{0}^{x} g(s) \, ds.
\]

Prove that for \( 1 < p < \infty \) the following estimate holds

\[
\|Tg\|_p \leq \frac{p}{p-1} \|g\|_p,
\]

and show that actually \( \|T\|_{L^p \to L^p} = \frac{p}{p-1} \). (Hint: Consider the integral \( \int_{0}^{x} f(s)s^\alpha s^{-\alpha} \, ds \) and apply Hölder with exponents \( p, p' \). Then minimise the quantity in \( \alpha \). For the sharpness consider the function \( f(x) = \chi_{[1, A]}(x)x^{-\frac{1}{p}} \), and let \( A \to \infty \))

Exercise 14:
For \( u \in S'(\mathbb{R}^d) \) and \( h \in S(\mathbb{R}^d) \) define the convolution \( h \ast u \) by

\[
\langle h \ast u, f \rangle = \langle u, \hat{h} \ast f \rangle,
\]

for a test function \( f \in S(\mathbb{R}^d) \), where \( \hat{h}(x) = h(-x) \).

In addition, define the support of the tempered distribution \( u \) to be the set

\[
\text{supp}(u) = \bigcap \left\{ \text{closed } K \subset \mathbb{R}^d : \phi \in S(\mathbb{R}^d), \supp(\phi) \subset K^c \implies \langle u, \phi \rangle = 0 \right\}
\]

- Show that for \( u = \delta_{x_0} \) and \( f \in S(\mathbb{R}^d) \) we have \( f \ast \delta_{x_0} \) is the function \( x \to f(x - x_0) \) and calculate \( \text{supp}(\delta_{x_0}) \).
Show that if $u \in S'(\mathbb{R}^d)$ and $\phi \in S(\mathbb{R}^d)$, then $\phi * u$ is a $C^\infty$ function. Moreover, for all multi-indices $\alpha$ there exists constants $C_\alpha, k_\alpha$ such that

$$|\partial^\alpha (\phi * u)(x)| \leq C_\alpha (1 + |x|)^{k_\alpha}.$$  

Furthermore, if $u$ has compact support, then $\phi * u$ is a Schwartz function. (Hint: At some point you have to interchange the distribution $u$ and an integral $\int_{\mathbb{R}^d}$)

Using the previous question (or any other possible way) show that if $T \in S'(\mathbb{R}^d)$ has compact support then $\hat{T} \in C^\infty$. 