Harmonic Analysis for Dispersive Equations

07. Problem Sheet

Exercise 21:
Let \( \phi \) be an integrable function on \( \mathbb{R}^d \) and set \( \phi_\epsilon(x) = \epsilon^{-d} \phi(x/\epsilon) \). Suppose that the least decreasing radial majorant of \( \phi \) is integrable, that is let

\[
\psi(x) = \sup_{|y| \geq |x|} |\phi(y)|,
\]
and suppose that \( \int_{\mathbb{R}^d} \psi(x)dx = A < \infty \). Prove that

- For every \( f \in L^p(\mathbb{R}^d) \), \( 1 \leq p \leq \infty \),

\[
\sup_{\epsilon > 0} |(f \ast \phi_\epsilon)(x)| \leq A M_f(x).
\]

- If \( \int_{\mathbb{R}^d} \phi(x)dx = 1 \) then

\[
\lim_{\epsilon \to 0} (f \ast \phi_\epsilon)(x) = f(x),
\]

for almost every \( x \in \mathbb{R}^d \).

(Hint: For the first question notice that it suffices to consider positive \( f \) and that trivially \( |f \ast \phi_\epsilon(x)| \leq f \ast \psi_\epsilon(x) \). Then observe that it suffices to consider only the case \( x = 0 \) and \( \epsilon = 1 \). Write the convolution \( f \ast \psi(0) \) using spherical coordinates and apply differentiation by parts once. For the second question follow the proof we did in class with the Lebesgue Differentiation Theorem)

Exercise 22:
Suppose \( u(x,y) \) is the Poisson integral of \( f \in L^p(\mathbb{R}^d) \), that is for \( x \in \mathbb{R}^d \) and \( y \in \mathbb{R}_+ \)

\[
u(x,y) = f \ast P_y(x),
\]

where

\[
P_y(x) = C_d \frac{y}{(|x|^2 + y^2)^\frac{d+1}{2}},
\]

and the constant is such that \( \int_{\mathbb{R}^d} P_y(x)dx = 1 \). Prove that there is a universal constant \( A > 0 \) such that

— Please turn the page! —
\[ \sup_{y>0} \left| y \frac{\partial u}{\partial x_j}(x, y) \right| \leq AMf(x). \]

(Hint: Use the previous exercise for \( \phi(x) = \frac{\partial}{\partial x_j} P_1(x) \))

**Exercise 23:**

Show that if \( T \) is a bounded operator on \( L^2(\mathbb{R}) \) that commutes with translations and dilations and anti-commutes with the reflection \( f(x) \to \tilde{f}(x) := f(-x) \), then \( T \) is a constant multiple of the Hilbert transform \( H \). (Hint: Write \( T \) is a multiplier operator with symbol \( u(\xi) \) and show that \( u(\xi) \) is a multiple of the function \( \text{sgn}(\xi) \) which is the symbol of the Hilbert transform \( H \))