

Harmonic Analysis for Dispersive Equations

07. Problem Sheet

Exercise 21:

Let ϕ be an integrable function on \mathbb{R}^d and set $\phi_\epsilon(x) = \epsilon^{-d}\phi(x/\epsilon)$. Suppose that the least decreasing radial majorant of ϕ is integrable, that is let

$$\psi(x) = \sup_{|y| \geq |x|} |\phi(y)|,$$

and suppose that $\int_{\mathbb{R}^d} \psi(x) dx = A < \infty$. Prove that

- For every $f \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$,

$$\sup_{\epsilon > 0} |(f * \phi_\epsilon)(x)| \leq A M f(x).$$

- If $\int_{\mathbb{R}^d} \phi(x) dx = 1$ then

$$\lim_{\epsilon \rightarrow 0} (f * \phi_\epsilon)(x) = f(x),$$

for almost every $x \in \mathbb{R}^d$.

(Hint: For the first question notice that it suffices to consider positive f and that trivially $|f * \phi_\epsilon(x)| \leq f * \psi_\epsilon(x)$. Then observe that it suffices to consider only the case $x = 0$ and $\epsilon = 1$. Write the convolution $f * \psi(0)$ using spherical coordinates and apply differentiation by parts once. For the second question follow the proof we did in class with the Lebesgue Differentiation Theorem)

Exercise 22:

Suppose $u(x, y)$ is the Poisson integral of $f \in L^p(\mathbb{R}^d)$, that is for $x \in \mathbb{R}^d$ and $y \in \mathbb{R}_+$

$$u(x, y) = f * P_y(x),$$

where

$$P_y(x) = C_d \frac{y}{(|x|^2 + y^2)^{\frac{d+1}{2}}},$$

and the constant is such that $\int_{\mathbb{R}^d} P_y(x) dx = 1$. Prove that there is a universal constant $A > 0$ such that

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$$\sup_{y>0} \left| y \frac{\partial u}{\partial x_j}(x, y) \right| \leq AMf(x).$$

(Hint: Use the previous exercise for $\phi(x) = \frac{\partial}{\partial x_j} P_1(x)$)

Exercise 23:

Show that if T is a bounded operator on $L^2(\mathbb{R})$ that commutes with translations and dilations and anti-commutes with the reflection $f(x) \rightarrow \tilde{f}(x) := f(-x)$, then T is a constant multiple of the Hilbert transform H . (Hint: Write T is a multiplier operator with symbol $u(\xi)$ and show that $u(\xi)$ is a multiple of the function $\operatorname{sgn}(\xi)$ which is the symbol of the Hilbert transform H)