$e := \sum_{n=0}^{\infty} \frac{1}{n!}$ <u>Satz 7.3</u> $\underbrace{e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n}$

<u>Beweis</u> $a_n := \left(1+\frac{1}{n}\right)^n \xRightarrow{\text{Binomialsatz}} a_n$

$$a_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{\overbrace{n(n-1)\dots(n-k+1)}^{\leq n \cdot n \dots n = n^k}}{k!} \frac{1}{n^k} \quad (1)$$

Also $a_n \leq \sum_{k=0}^{n} \frac{1}{k!} \leq e.$

Also $\limsup a_n \leq e$ (2)

Andererseits, für $m \leq n$, m fest gilt

$$a_n \geq \sum_{k=0}^{m} \underbrace{\frac{n(n-1)\dots(n-k+1)}{k!} \frac{1}{n^k}} =$$

$$= \frac{1}{k!} \frac{n(n-1)\dots(n-k+1)}{n^k} = \frac{1}{k!} 1\left(1-\frac{1}{n}\right)\dots\left(1-\frac{k-1}{n}\right)$$

$\xrightarrow{n\to\infty} \frac{1}{k!} \cdot 1 \cdot 1 \cdot \ldots \cdot 1 = \frac{1}{k!}$

Also $a_n \geq \sum_{k=0}^{m} \frac{n(n-1)\ldots(n-k+1)}{k!} \frac{1}{k!} \xrightarrow{n\to\infty} \sum_{k=0}^{m} \frac{1}{k!}$

Also $\liminf a_n \geq \sum_{k=0}^{m} \frac{1}{k!} \quad \forall m \in \mathbb{N}$

$\Longrightarrow \liminf a_n \geq \sum_{k=0}^{\infty} \frac{1}{k!} = e \quad (3)$

Also

$\limsup a_n \overset{(2)}{\leq} e \overset{(3)}{\leq} \liminf a_n \leq \limsup a_n$ gilt immer

$\Longrightarrow \liminf a_n = \limsup a_n = e$

$\Longrightarrow \lim_{n\to\infty} a_n$ existiert

und $\lim_{n\to\infty} a_n = e$