Heute: sin, cos, tan und arccos, arcsin, arctan.
und Anwendung in Polardarstellung

Eigenschaften von $\sin x, \cos x$:

$\forall x \in \mathbb{R}$ gilt (i) $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$

(ii) $\sin(x+\pi) = -\sin x$, $\cos(x+\pi) = -\cos x$.

(iii) $\sin(x+2\pi) = \sin x$, $\cos(x+2\pi) = \cos x$.

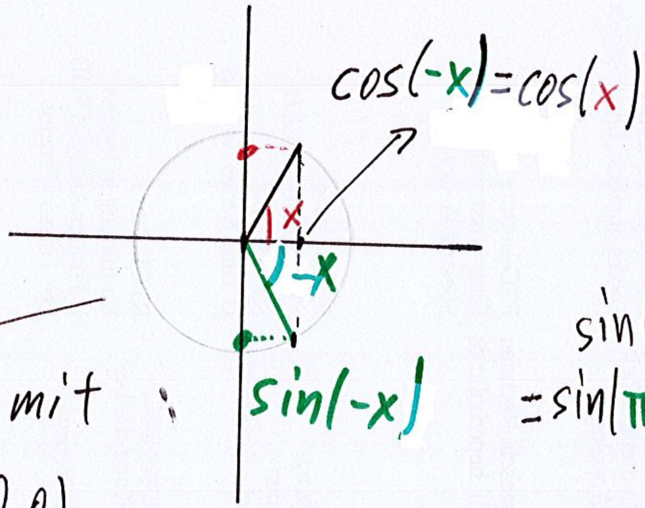
(d.h. sin, cos sind 2π-periodisch).

(iv) $\sin(\pi - x) = \sin x$, $\cos(\pi - x) = -\cos x$

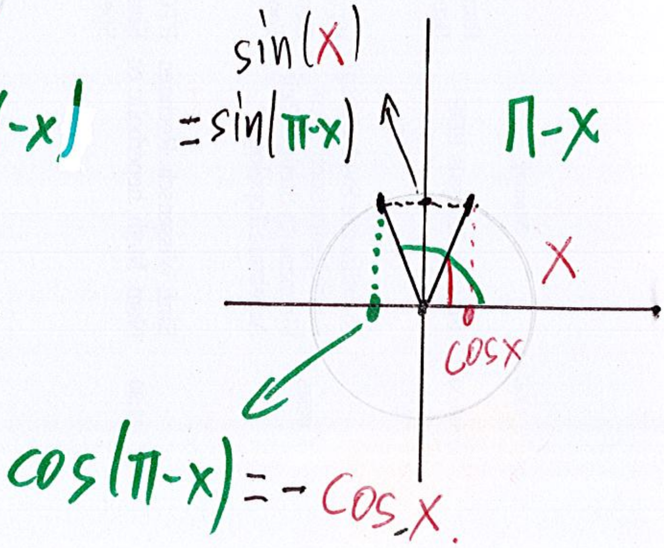
(v) $\sin\left(x + \frac{\pi}{2}\right) = \cos x$, $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$

(vi) $\forall k \in \mathbb{Z}$ gilt $\cos(k\pi) = (-1)^k$, $\sin(k\pi) = 0$.

Illustration von (i), (iv).



Einheitskreis mit Zentrum (0,0).



$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

Nullstellen von sin, cos: $\forall x \in \mathbb{R}$ gilt

$\cos x = 0 \iff \exists\, k \in \mathbb{Z}$ mit $x = (2k+1)\frac{\pi}{2}$

$\sin x = 0 \iff \exists\, k \in \mathbb{Z}$ mit $x = k\pi$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin x$	$\frac{\sqrt{0}}{2} = 0$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2} = 1$
$\cos x$	$\frac{\sqrt{4}}{2} = 1$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{0}}{2} = 0$

Die Zahlen $\cos\left(\frac{437\pi}{6}\right)$, $\sin\left(\frac{437\pi}{6}\right)$ sind gleich (Bsp 9.8)

(1) $\frac{\sqrt{3}}{2}, \frac{1}{2}$, (2) $-\frac{\sqrt{3}}{2}, \frac{1}{2}$ (3) $\frac{\sqrt{3}}{2}, -\frac{1}{2}$, (4) $-\frac{\sqrt{3}}{2}, -\frac{1}{2}$.

(5) $\frac{1}{2}, \frac{\sqrt{3}}{2}$, (6) $-\frac{1}{2}, \frac{\sqrt{3}}{2}$. (7) $\frac{1}{2}, -\frac{\sqrt{3}}{2}$, (8) $-\frac{1}{2}, -\frac{\sqrt{3}}{2}$

(9) keine der obigen Antworten

(10) keine Ahnung.

Lö: $\frac{437\pi}{6} = \frac{437}{12} 2\pi$

$$\begin{array}{r|l} 437 & 12 \\ -36 & \overline{36} \\ \hline 77 & \\ -72 & \\ \hline 5 & \end{array}$$

Also $437 = 36 \cdot 12 + 5$

$\Rightarrow \frac{437}{12} 2\pi = 36 \cdot 2\pi + \frac{5}{12} 2\pi$.

Also $\cos\left(\frac{437}{6}\pi\right) = \cos\left(36 \cdot 2\pi + \frac{5\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right)$.

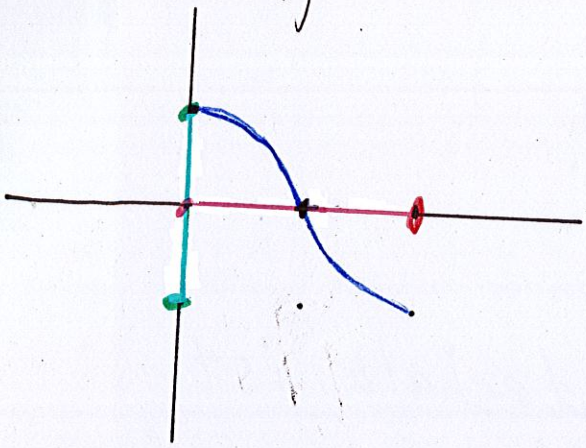
ähnlich $\sin\left(\frac{437\pi}{6}\right) = \sin\left(\frac{5\pi}{6}\right)$.

Aber $\cos\left(\frac{5\pi}{6}\right) = \cos\left(\pi - \frac{\pi}{6}\right) \overset{(iv)}{=} -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$.

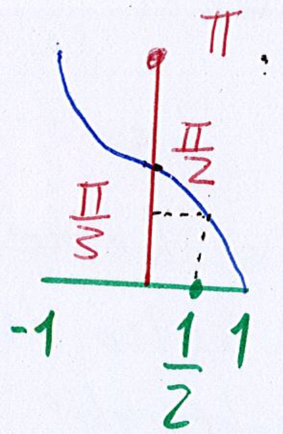
und $\sin\left(\frac{5\pi}{6}\right) = \sin\left(\pi - \frac{\pi}{6}\right) \overset{(iv)}{=} \sin\frac{\pi}{6} = \frac{1}{2}$.

Also ist (2) die richtige Antwort.

cus cosinus: $\cos : \mathbb{R} \to [-1,1]$ ist nicht injektiv. Aber:



$\cos : [0,\pi] \to [-1,1]$ ist bijektiv, stetig und streng monoton fallend. Ebenso ist die Umkehrabbildung $\arccos : [-1,1] \to [0,\pi]$.



Bsp 9.9 $\arccos\left(\frac{1}{2}\right) = ?$

Welcher Winkel in $[0,\pi]$ hat $\cos = \frac{1}{2}$?

Antwort: $\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$ da $\frac{\pi}{3} \in [0,\pi]$ und $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

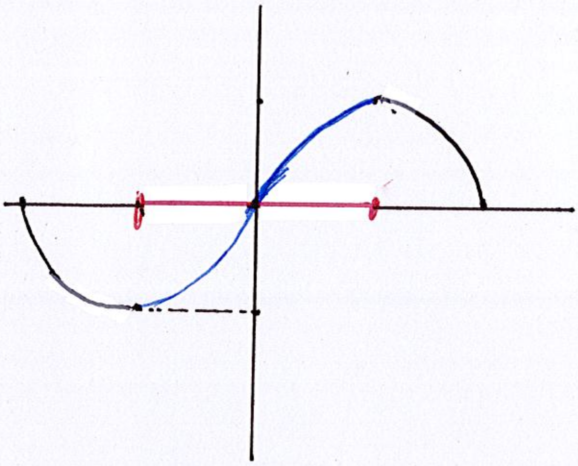
Ähnlich: $\arccos(0) = \frac{\pi}{2}$, $\arccos(1) = 0$, $\arccos(-1) = \pi$

$\left(\cos\left(\frac{\pi}{2}\right) = 0, \quad \cos(0) = 1, \quad \cos(\pi) = -1 \right)$

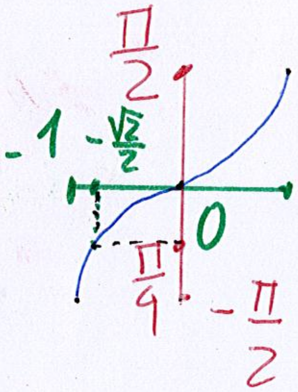
Achtung: $\cos\left(\frac{7\pi}{3}\right) = \frac{1}{2}$ aber $\arccos\left(\frac{1}{2}\right) \neq \frac{7\pi}{3}$

da $\frac{7\pi}{3} \notin [0,\pi]$.

Arcussinus: $\sin: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$



ist bijektiv, stetig und streng monoton wachsend und ebenso ihre Umkehrabbildung.



$\arcsin: [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Bsp 9.10

$\arcsin(0) = 0 \quad (\sin(0) = 0)$

$\arcsin(1) = \frac{\pi}{2} \quad \left(\sin\left(\frac{\pi}{2}\right) = 1\right)$

$\arcsin(-1) = -\frac{\pi}{2} \quad \left(\sin\left(-\frac{\pi}{2}\right) = -1\right)$

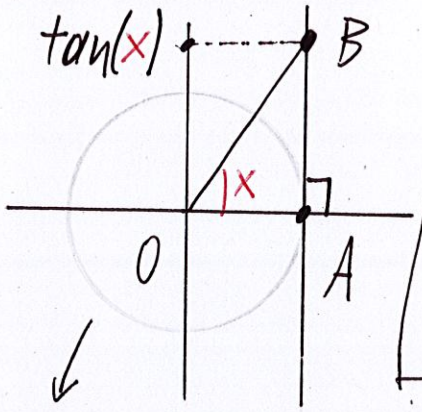
$\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4} \quad \left(\sin\left(-\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}\right)$.

Tipp:

arccos y	arcsin z	x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
	z	sin x	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
y		cos x	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Die Funktion $\tan: \mathbb{R} \setminus A \to \mathbb{R}$

$x \mapsto \tan x = \frac{\sin x}{\cos x}$, wobei $A = \left\{ \frac{\pi}{2} + k\pi : k \in \mathbb{Z} \right\}$ (Nullstellen von $\cos$) heißt <u>Tangens</u>.

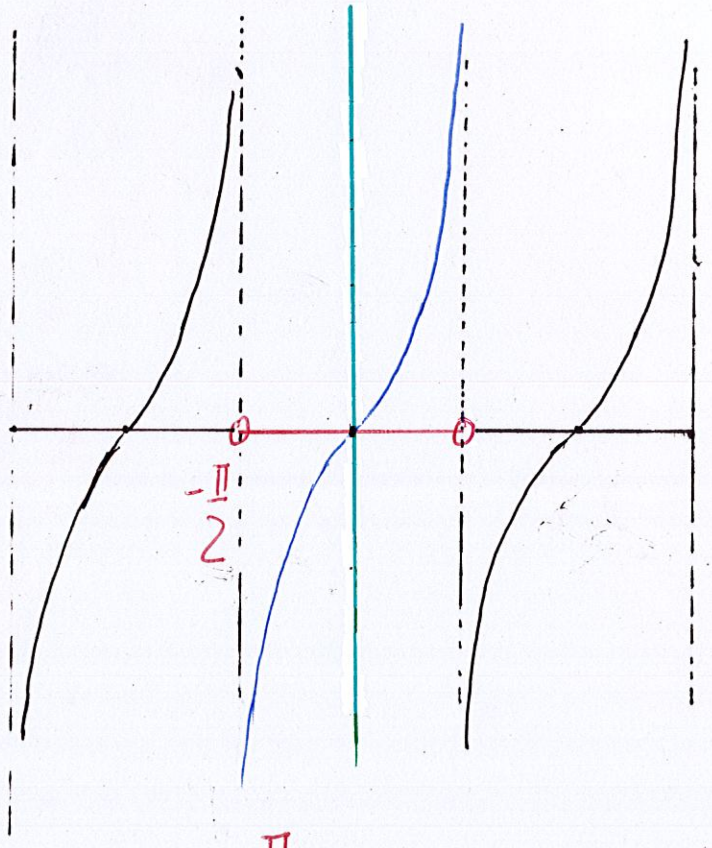


Einheitskreis $|OA| = 1 \; (*)$

$x \in \left(0, \frac{\pi}{2}\right) \Rightarrow \tan x = \frac{|AB|}{|OA|} \overset{(*)}{=} |AB|$

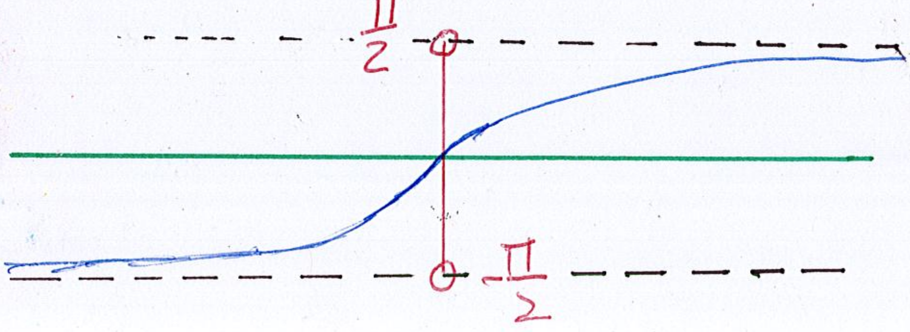
Es gilt auch $\lim\limits_{x \to \frac{\pi}{2}-} \tan x = \infty$

$\lim\limits_{x \to -\frac{\pi}{2}+} \tan x = -\infty$.



$\tan: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$

Ist streng monoton wachsend stetig und bijektiv und ebenso ihre Umkehrabbildung.



$\arctan: \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Bsp 9.10. $\arctan(0)=0$ $\quad(\tan 0 = 0)$.

$\arctan(\sqrt{3})=\frac{\pi}{3}$ $\quad(\tan\frac{\pi}{3}=\sqrt{3})$.

$\lim\limits_{y\to\infty}\arctan(y)=\frac{\pi}{2}$ $\quad(\lim\limits_{x\to\frac{\pi}{2}-}\tan(x)=\infty)$.

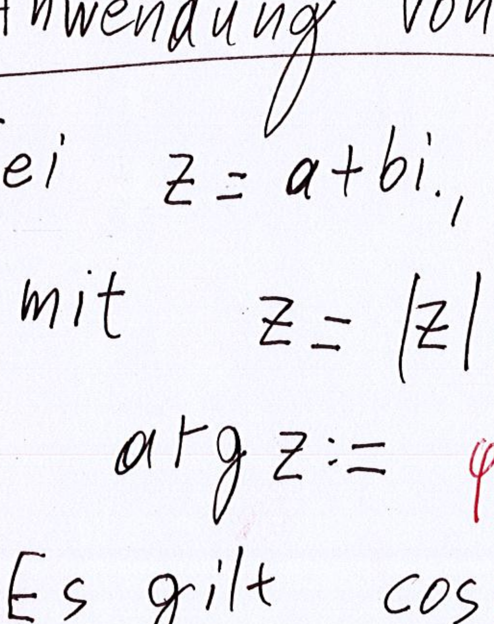
$\lim\limits_{y\to-\infty}\arctan(y)=-\frac{\pi}{2}$ $\quad(\lim\limits_{x\to-\frac{\pi}{2}+}\tan(x)=-\infty)$

Anwendung von arccos in der Polardarstellung

Sei $z=a+bi$, $a,b\in\mathbb{R}$. Dann $\exists\ \varphi\in(-\pi,\pi]$ mit $z=|z|(\cos\varphi+i\sin\varphi)=|z|e^{i\varphi}$

$\arg z := \varphi$ (Argument von z).

Es gilt $\cos\varphi=\frac{a}{|z|}=\frac{a}{\sqrt{a^2+b^2}}$ aber

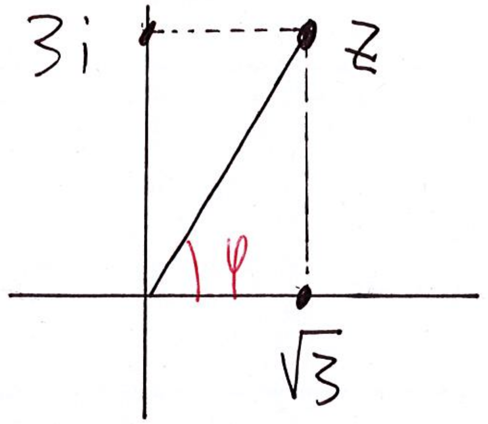


NICHT immer $\varphi=\arccos\left(\frac{a}{\sqrt{a^2+b^2}}\right)$

da $\arccos([-1,1])=[0,\pi]$

aber φ nicht immer in $[0,\pi]$.

<u>Fall 1</u> $b \geq 0$ dann $\varphi \in [0, \pi]$ also tatsächlich $\varphi = \arccos\left(\frac{a}{\sqrt{a^2+b^2}}\right)$.



Bsp 9.11: $z = \sqrt{3} + 3i$.

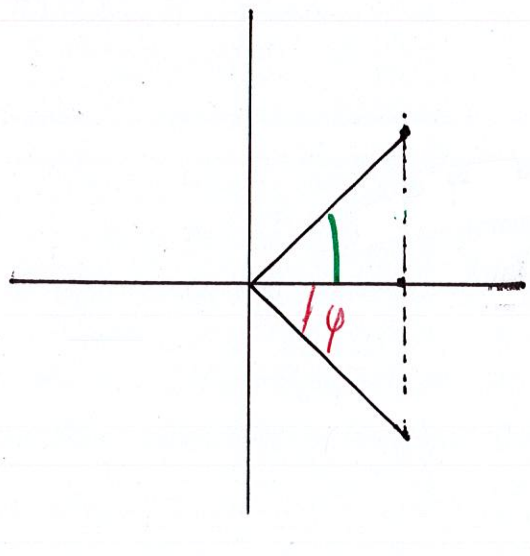
$3 \geq 0$ also

$\varphi = \arccos\left(\frac{\sqrt{3}}{\sqrt{\sqrt{3}^2+3^2}}\right) =$

$= \arccos\left(\frac{\sqrt{3}}{\sqrt{12}}\right) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

<u>Fall</u> $b < 0$. Dann $\varphi \in (-\pi, 0)$ und

$\varphi = -\arccos\left(\frac{a}{\sqrt{a^2+b^2}}\right)$.



Bsp 9.12: $z = 1 - i$.

$-1 < 0$, also

$\varphi = -\arccos\left(\frac{1}{\sqrt{1^2+(-1)^2}}\right)$

$= -\arccos\left(\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$.